# Black hole radiation phase space from 1d higher point of view

arXiv:2209.04548

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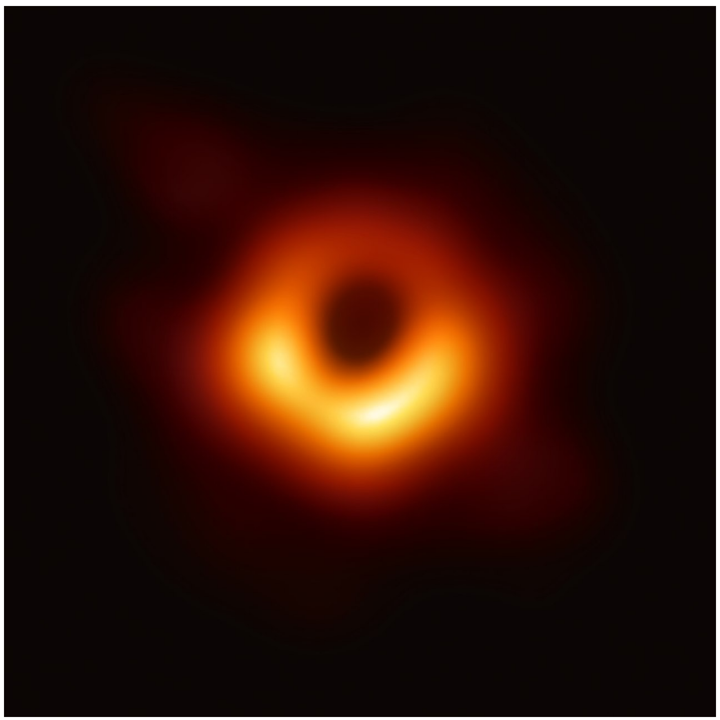
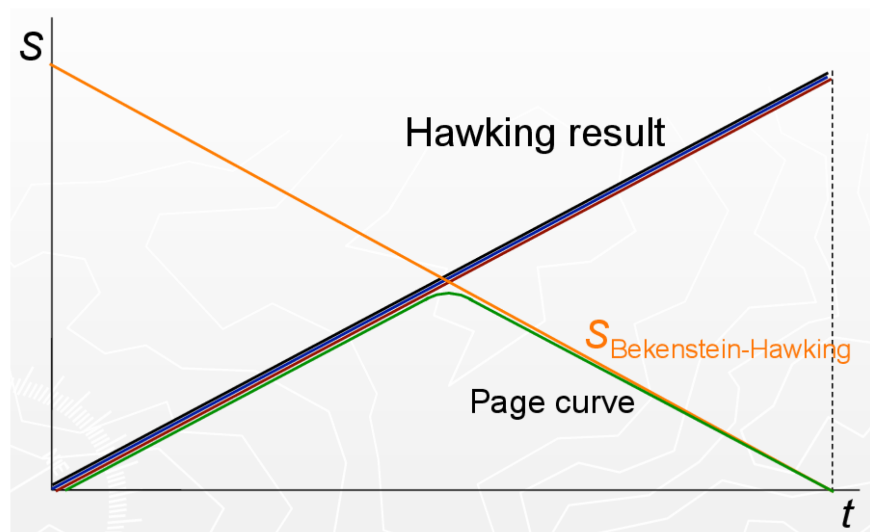
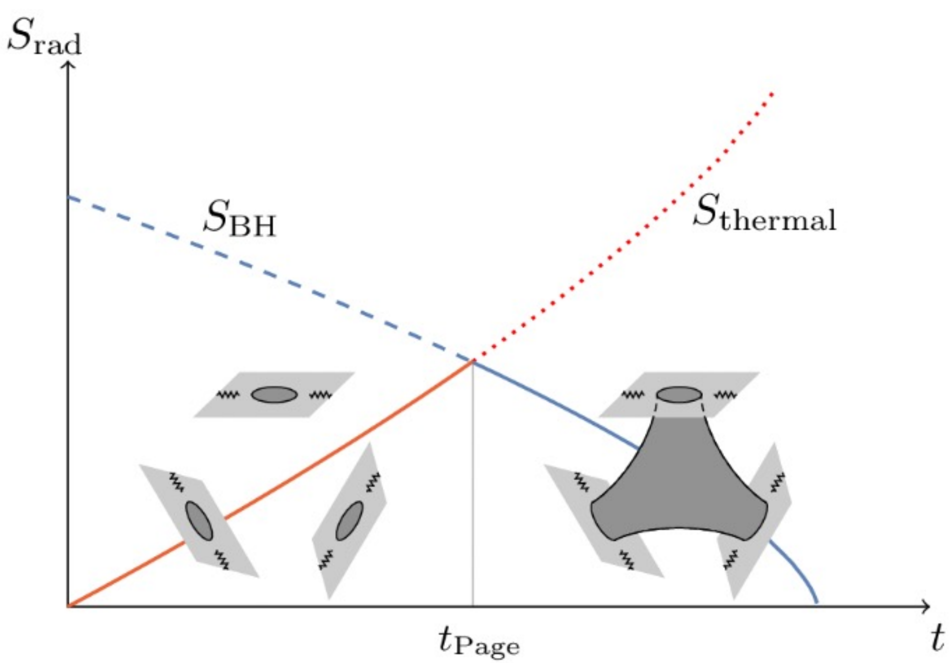
# Information loss Paradox of Black Holes and Page curve



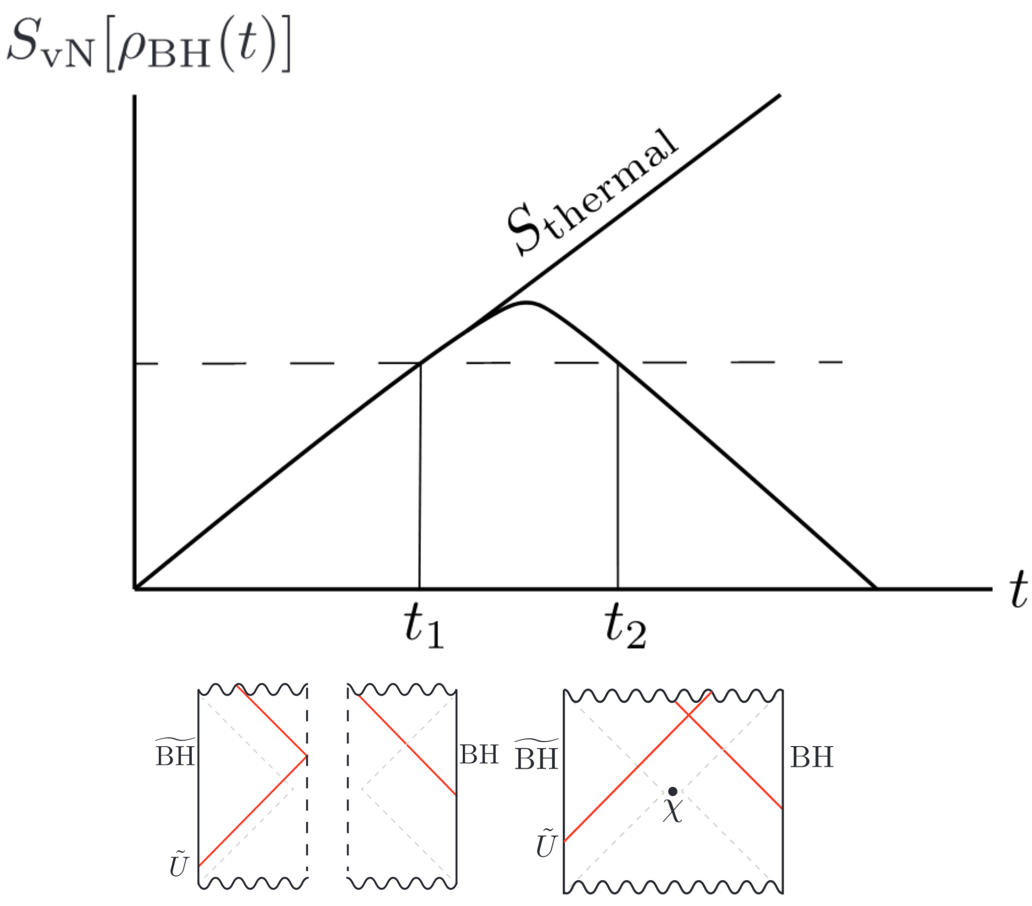


# Page curve and wormholes




In **2d JT theory**, the transition between disconnected and connected topologies in the Euclidean path integral gives rise to the the Page curve for the entropy of Hakwking radiation.

# Page curve and connectivity



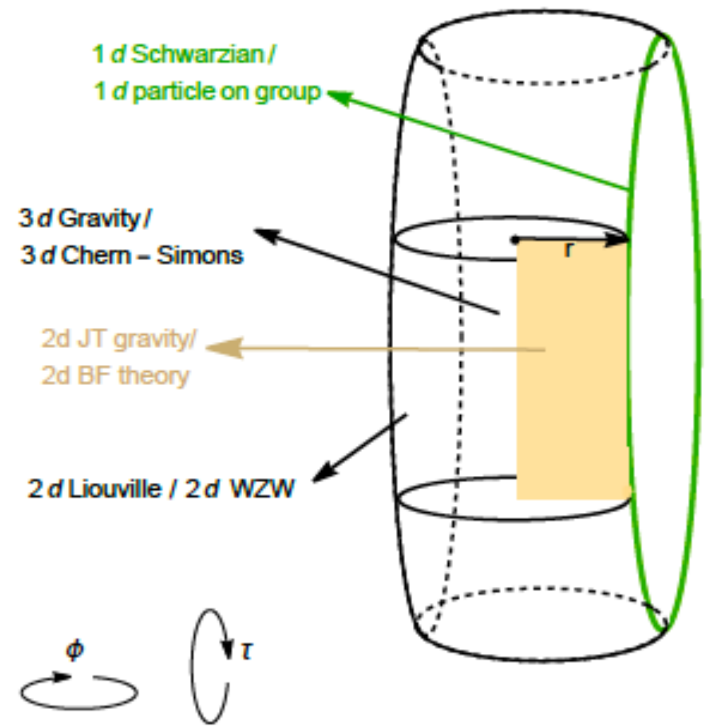

Entanglement entropy is not enough to determine the connectivity of spacetime, but other parameters such as Berry phase can determine the non-factorization quality of Hilbert space and detects the presence of the wormholes.

**Effects of higher d?**

# How to extend the story?

Three ways to extend this story:

- Going to **higher dimensions**.
- Extending to **other theories** than JT such as **BTZ, BF, Chern-Simons, Liouville**, and finding the connections between them.
- Using other measures of correlations than entanglement enropy, specifically those of ***mixed*** correlation, such as *MI, LN, EoP, CoP, reflected entropy*, etc.



# JT Gravity

$$I_{JT} = \frac{1}{16\pi G}\int d^2x\sqrt{-g}\left[\phi_0 R + \phi\left(R + \frac{2}{L^2}\right)\right] + \frac{1}{8\pi G}\int dt\sqrt{-\gamma}\left[\phi_0 K + \phi\left(K - \frac{1}{L}\right)\right].$$

For simulating black hole d.o.f, one can use:

$$I = I_{JT} + \mu\int_{\text{brane}} dy,$$

$$I_{JT} = -\frac{S_0}{2\pi}\left[\frac{1}{2}\int_{\mathcal{M}} R + \int_{\partial\mathcal{M}} K\right] - \left[\frac{1}{2}\int_{\mathcal{M}}\phi(R+2) + \int_{\partial\mathcal{M}}\phi K\right]$$

# EOW brane and mixed systems

The mixed state $\rho_{AB} = \sum_{i=1}^{k} p_i \rho_A^i \otimes \rho_B^i, \quad \sum_{i=1}^{k} p_i = 1$


Defined on the Hilbert space: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

The entangled states of BH (B) and Radiation (R) $|\Psi\rangle = \frac{1}{\sqrt{k}} \sum_{i=1}^{k} |\psi_i\rangle_B |i\rangle_R$

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Don, McBride, Weng

Density matrix of the subsystem R $\rho_R = \frac{1}{k} \sum_{i,j=1}^{k} |j\rangle \langle i|_R \langle \psi_i|\psi_j\rangle_B$

The EOW has "*k*" orthonormal states or flavors. By increasing *k*, later regimes of an evaporating BH can be probed.

# Negativity as a measure of mixed correlation

A bipartite mixed state: $\mathcal{H}_R = \mathcal{H}_{R_1} \otimes \mathcal{H}_{R_2}$ $k_1$ and $k_2$ states $k = k_1 k_2$

Density matrix
$$\rho_{R_1R_2} = \frac{1}{k} \sum_{i_1,i_2=1}^{k_1} \sum_{j_1,j_2=1}^{k_2} |i_1, j_1\rangle\langle i_2, j_2| \langle \psi_{i_2,j_2}|\psi_{i_1,j_1}\rangle$$

Partially transposed density matrix
$$\rho_{R_1R_2}^{T_{R_2}} = \frac{1}{k} \sum_{i_1,i_2=1}^{k_1} \sum_{j_1,j_2=1}^{k_2} |i_1, j_2\rangle\langle i_2, j_1| \langle \psi_{i_2,j_2}|\psi_{i_1,j_1}\rangle.$$

**Negativity**
$$\mathcal{N}(\rho_{AB}) = \frac{||\rho_{AB}^{T_B}||_1 - 1}{2} = \sum_i \frac{|\lambda_i| - \lambda_i}{2} = \sum_{i:\lambda_i<0} |\lambda_i|.$$

**Logarithmic Negativity**
$$\mathcal{E}(\rho_{AB}) = \log\left(\sum_i |\lambda_i|\right) = \log\left(2\mathcal{N}(\rho_{AB}) + 1\right)$$

# Partition Function and Renyi Negativity

$$Z_{\text{saddle}} = Z_{\text{grav}} f(k_1, k_2) \qquad Z_n \sim e^{S_0 \chi}.$$

For a disk connecting "n" boundaries in JT gravity, we have:

$$Z_n = e^{S_0} \int_0^{\infty} ds \rho(s) y(s)^n, \quad y(s) \equiv e^{-\frac{\beta s^2}{2}} 2^{1-2\mu} \left| \Gamma\left(\mu - \frac{1}{2} + is\right) \right|^2$$

$$\rho(s) = \frac{s}{2\pi^2} \sinh(2\pi s)$$

← Disk density of states

$\chi(g)$ Number of disjoint cycles of permutations

$$\begin{aligned} \text{Tr}\left[(\rho_R^{T_2})^n\right] &= \frac{1}{(kZ_1)^n} \sum_{g \in S_n} \left( \prod_{i=1}^{\chi(g)} Z_{|c_i(g)|} \right) k_1^{\chi(g^{-1}X)} k_2^{\chi(g^{-1}X^{-1})} \\ &\sim \frac{1}{(ke^{S_0})^n} \sum_{g \in S_n} (e^{S_0})^{\chi(g)} k_1^{\chi(g^{-1}X)} k_2^{\chi(g^{-1}X^{-1})}, \end{aligned}$$

# Saddles from Negativity

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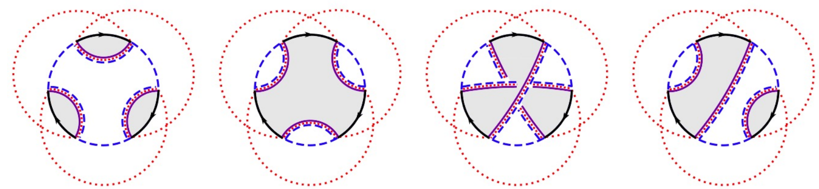
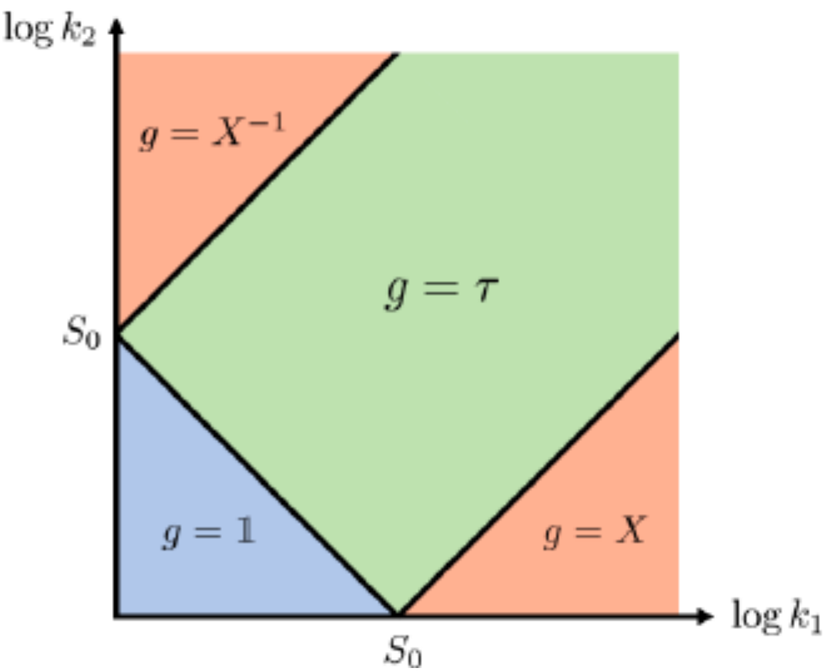
| | | | |
|---|---|---|---|
| Totally disconnected: | $e^{S_0} \gg k_1 k_2$ | $\rightarrow$ | $g = \mathbb{1}$ |
| Cyclically connected: | $k_1 \gg k_2 e^{S_0}$ | $\rightarrow$ | $g = X$ |
| Anti-cyclically connected: | $k_2 \gg k_1 e^{S_0}$ | $\rightarrow$ | $g = X^{-1}$ |
| Pairwise connected: | $k_1 k_2 \gg e^{S_0}$, $e^{-S_0} \ll k_1/k_2 \ll e^{S_0}$ | $\rightarrow$ | $g = \tau$ |

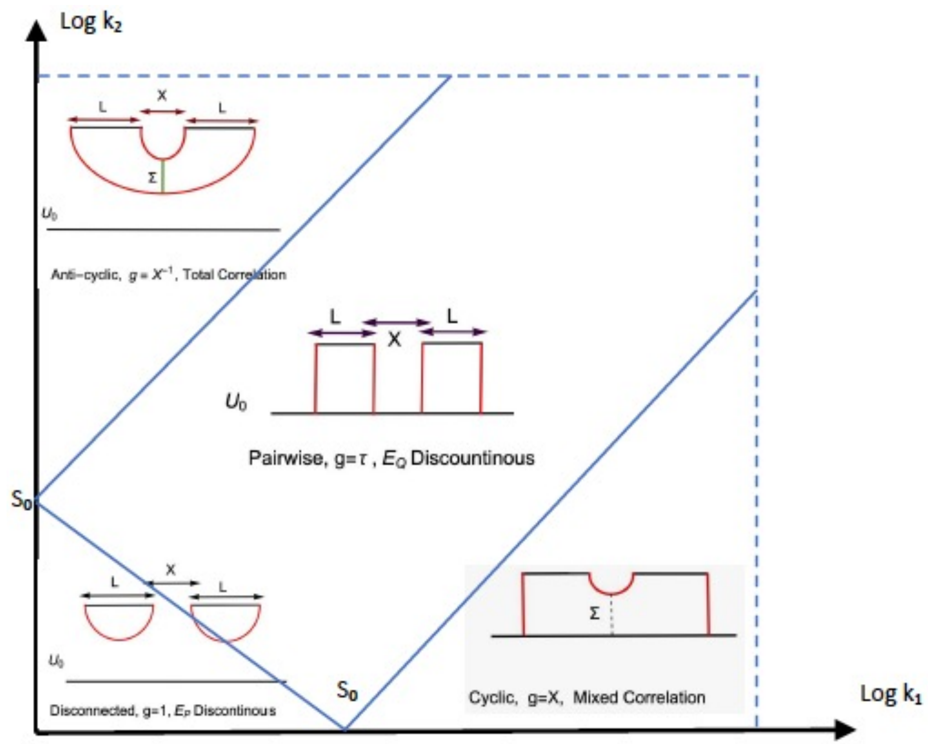





Negativities in Dominant Phases

| $g$ | $\mathbb{1}$ | $X$ | $X^{-1}$ | $\tau$ |
|---|---|---|---|---|
| $\mathcal{N}_{2m}^{(\text{even})}$ | $\frac{1}{k^{2m-1}}$ | $\frac{Z_{2m}}{k_2^{2m-2} Z_1^{2m}}$ | $\frac{Z_{2m}}{k_1^{2m-2} Z_1^{2m}}$ | $\frac{C_m Z_2^m}{k^{m-1} Z_1^{2m}}$ |
| $\mathcal{N}_{2m-1}^{(\text{odd})}$ | $\frac{1}{k^{2m-2}}$ | $\frac{Z_{2m-1}}{k_2^{2m-2} Z_1^{2m-1}}$ | $\frac{Z_{2m-1}}{k_1^{2m-2} Z_1^{2m-1}}$ | $\frac{(2m-1) C_{m-1} Z_2^{m-1}}{k^{m-1} Z_1^{2m-2}}$ |
| $\mathcal{E}$ | $0$ | $\log k_2$ | $\log k_1$ | $\frac{1}{2}(\log k - S_0) + \log \frac{8}{3\pi}$ |
| $S^{T_2}$ | $\log k$ | $\log k_2 + S_0$ | $\log k_1 + S_0$ | $\frac{1}{2}(\log k + S_0) - \frac{1}{2}$ |
| $S^{T_2(2)}$ | $\log k$ | $2 \log k_2 + S_0$ | $2 \log k_1 + S_0$ | $\log k - \frac{1}{2}$ |

# Negativity versus Mutual Information



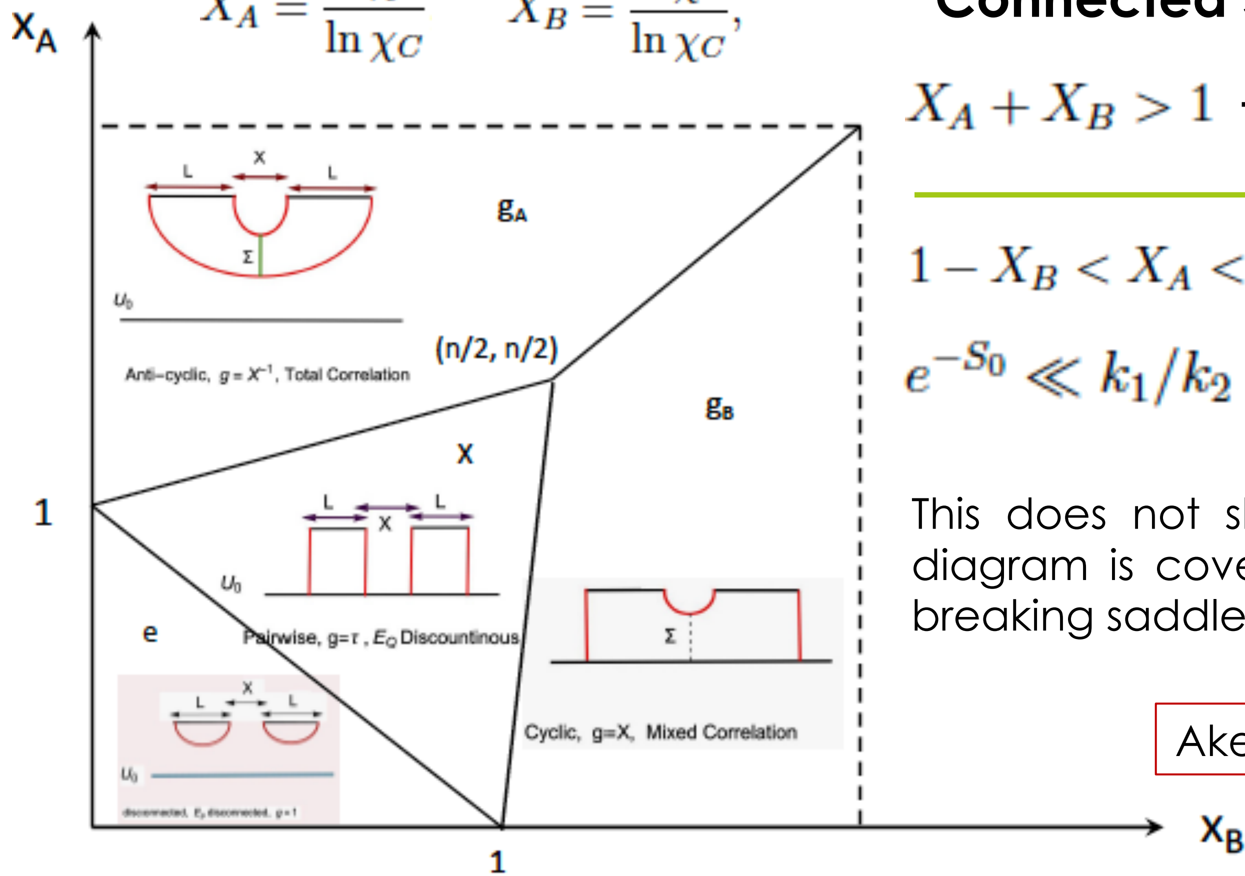

**Our first result:**

Most of the phase space is covered by the saddle which breaks the replica symmetry. This result is both for 2d JT gravity and 10d supergravity models.

$$E_N(\rho_{AB}) \simeq \frac{1}{2} I(A:B) + \log \frac{8}{3\pi},$$

# Page curve for reflected entropy?

$$X_A = \frac{\ln \chi_A}{\ln \chi_C} \qquad X_B = \frac{\ln \chi_B}{\ln \chi_C},$$




**Connected Saddle:**

$X_A + X_B > 1 \rightarrow k_1 k_2 \gg e^{S_0}$ Pairwise?

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$1 - X_B < X_A < 1 + X_B,$ →

$e^{-S_0} \ll k_1/k_2 \ll e^{S_0}$ Cyclicly connected?

This does not show that most of the phase diagram is covered by the replica symmetry breaking saddle, and therefore is *wrong*!!

Akers, Faulkner, Lin, Rath

# Why reflected entropy gave worng phase diagram?

Phase space can be probed by either D(distance between strips), *L* (width of the strips), and $U_0$ the hard wall location; or by $k_1$, $k_2$, $S_0$, or by the three bond dimensions, or horizon area of wormholes, or by $S^{(eq)}{}_{A1, A, B}$, or by $V_A$, $V_{A1}$, $V_B$.

As observed from various mixed correlation measures, the phase that breaks the replica symmetry should cover most of the phase space, but that is not the case for reflected entropy, why?

Recently, in **2302.10208**, ***Hayden, Lemm, Sorce***, "Reflected entropy is not a correlation measure", it has been shown that reflected entropy is not monotonically decreasing under partial trace and therefore **it cannot be a measure of physical correlation**!!

# Dimension reduction from Einstein to JT

3d Einstein:

$$S = \frac{1}{16\pi G^{(3)}} \int d^3x \sqrt{-g}(R^{(3)} - 2\Lambda)$$

2d JT:

$$S = \frac{1}{16\pi G} \left[ \int d^2x \sqrt{-g} \Phi_0 R + \int d^2x \sqrt{-g} \Phi \left( R + \frac{2}{\ell^2} \right) \right] + S_{\text{matter}}$$

The metric field is independent of $\boldsymbol{\varphi}$

$$ds^2 = g_{\mu\nu}\, \mathrm{d}x^\mu \, \mathrm{d}x^\nu = h_{ij}(x^i)\, \mathrm{d}x^i \, \mathrm{d}x^j + \phi^2(x^i)\ell_3^2 \, \mathrm{d}\varphi^2$$

Then, the action reduces to: $S = \frac{2\pi\alpha\ell_3}{16\pi G^{(3)}} \int d^2x \sqrt{-h}\phi(R^{(2)} - 2\Lambda) \qquad \alpha \in (0, 1]$

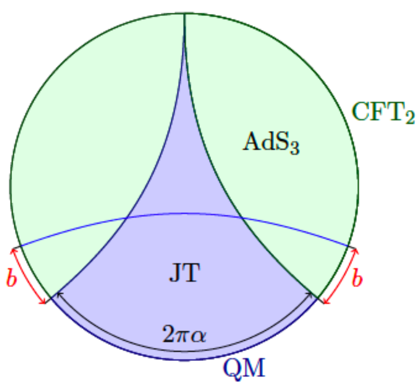
$$\ell_3 = \ell \qquad G^{(3)} = \ell G^{(2)} \quad \rightarrow \quad \Phi = 2\pi\alpha\phi$$

$$\Phi|_{bdy} = \frac{\Phi_r}{\epsilon} \qquad \phi|_{bdy} = \frac{\ell}{\epsilon} \qquad \Phi = \Phi_r \frac{\sqrt{g_{\varphi\varphi}}}{\ell^2} = \Phi_r \frac{\phi}{\ell}$$

Evita Verheijden and Erik Verlinde

# Partial reduction over $2\pi\alpha$ and EE

Extremal $AdS_2$ BH

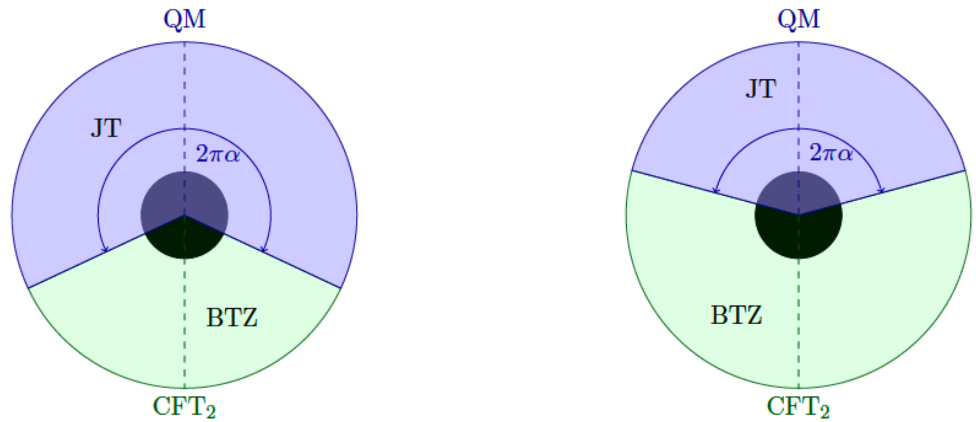



$$S = \frac{1}{2G} \operatorname{arcsinh} \frac{r\Delta\varphi}{2\ell} \qquad G = G^{(2)}$$

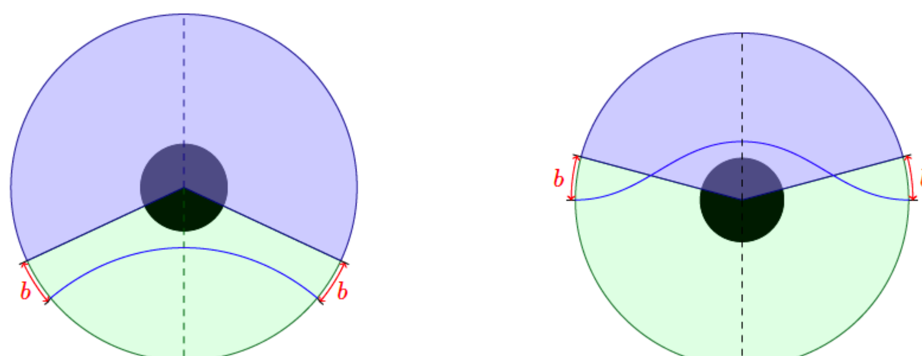
$$S = \frac{1}{4G}\left(2\log\frac{\Phi_r + 2b}{\ell}\right)$$

Verheijden, Verlinde

$$\Phi_r = 2\pi\ell\alpha$$

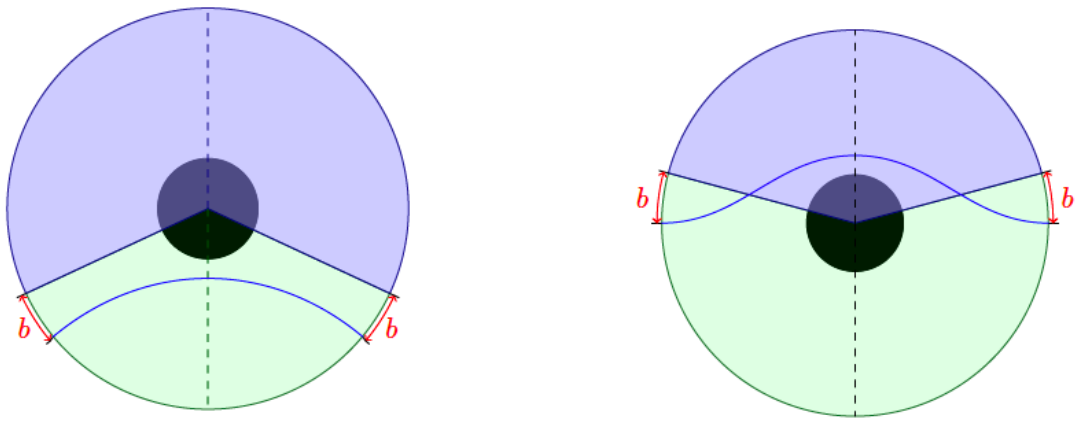



(a) $t < t_{\mathrm{Page}}$ (b) $t > t_{\mathrm{Page}}$




(a) $t < t_{\mathrm{Page}}$ (b) $t > t_{\mathrm{Page}}$

# BTZ EE from 3d point of view




(a) $t < t_{\text{Page}}$ (b) $t > t_{\text{Page}}$

By decreasing $\alpha$ from 1 to 0, the BH geometrically evaporates.

$$S = \frac{1}{2G}\,\text{arcsinh}\frac{2\pi\ell^2 r}{\beta}\sinh\frac{\pi}{\beta}\ell\Delta\varphi$$

$$S = \frac{1}{4G}\left(2\log\sinh\frac{\pi}{\beta}\left(2\pi\ell(1-\alpha)-2b\right)\right)$$

$$S = \frac{1}{4G}\left(2\log\sinh\frac{\pi}{\beta}\left(2\pi\ell\alpha+2b\right)\right)$$

Verheijden, Verlinde (VV) setup

# Now consider two intervals

Two intervals with radial angles "**μ**" and with radial distance **ν** between them where **α**=2**μ**+**ν**. Then the MI would be:

**Our 2nd result**

$$I(A:B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$$


Before Page Time we get:

$$\sinh^2 \frac{\pi}{\beta}\Big(2\pi\ell(1-\mu) - 2b_c\Big) = \sinh \frac{\pi}{\beta}\Big(2\pi\ell(1-\nu) - 2b_c\Big) \sinh \frac{\pi}{\beta}\Big(2\pi\ell(1-2\mu-\nu) - 2b_c\Big),$$

After Page Time we get:

$$\sinh^2 \frac{\pi}{\beta}\Big(2\pi\ell\mu + 2b_c\Big) = \sinh \frac{\pi}{\beta}\Big(2\pi\ell(2\mu+\nu) + 2b_c\Big) \sinh \frac{\pi}{\beta}\Big(2\pi\ell\nu + 2b_c\Big),$$

# Before the Page Time

$$\sinh^2 \frac{\pi}{\beta}\Big(2\pi\ell(1-\mu)-2b_c\Big) = \sinh\frac{\pi}{\beta}\Big(2\pi\ell(1-\nu)-2b_c\Big)\sinh\frac{\pi}{\beta}\Big(2\pi\ell(1-2\mu-\nu)-2b_c\Big),$$

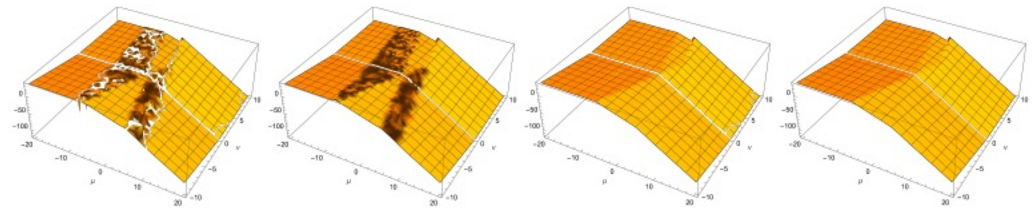
$$b_c(1) \to \frac{\beta}{4\pi}\left(\log(2) - 4i\pi c_1 - 2\log\left(-\sqrt{\frac{e^{-\frac{4\pi^2 l}{\beta}}\left(\xi e^{\frac{2\pi^2 l(4\mu+\nu)}{\beta}} + 2e^{\frac{4\pi^2 l\mu}{\beta}} - e^{\frac{8\pi^2 l\mu}{\beta}} - 1\right)}{1-e^{-\frac{4\pi^2 l\nu}{\beta}}}}\right)\right), \quad (4.5)$$

$$b_c(2) \to \frac{\beta}{4\pi}\left(\log(2) - 4i\pi c_1 - \log\left(\frac{e^{-\frac{4\pi^2 l}{\beta}}\left(\xi e^{\frac{2\pi^2 l(4\mu+\nu)}{\beta}} + 2e^{\frac{4\pi^2 l\mu}{\beta}} - e^{\frac{8\pi^2 l\mu}{\beta}} - 1\right)}{1-e^{-\frac{4\pi^2 l\nu}{\beta}}}\right)\right), \quad (4.6)$$

$$b_c(3) \to \frac{\beta}{4\pi}\left(\log(2) - 4i\pi c_1 - 2\log\left(-\sqrt{\frac{e^{-\frac{4\pi^2 l}{\beta}}\left(\xi\left(-e^{\frac{2\pi^2 l(4\mu+\nu)}{\beta}}\right) + 2e^{\frac{4\pi^2 l\mu}{\beta}} - e^{\frac{8\pi^2 l\mu}{\beta}} - 1\right)}{1-e^{-\frac{4\pi^2 l\nu}{\beta}}}}\right)\right), \quad (4.7)$$

$$b_c(4) \to \frac{\beta}{4\pi}\left(\log(2) - 4i\pi c_1 - \log\left(\frac{e^{-\frac{4\pi^2 l}{\beta}}\left(\xi\left(-e^{\frac{2\pi^2 l(4\mu+\nu)}{\beta}}\right) + 2e^{\frac{4\pi^2 l\mu}{\beta}} - e^{\frac{8\pi^2 l\mu}{\beta}} - 1\right)}{1-e^{-\frac{4\pi^2 l\nu}{\beta}}}\right)\right), \quad (4.8)$$

$$T_H = \frac{1}{\pi}\sqrt{\frac{8\pi G E}{2\Phi_r}}$$



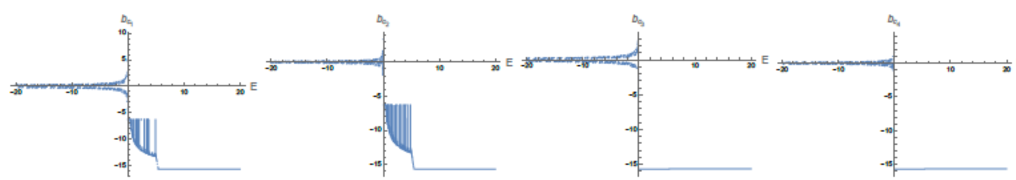


*Figure 6: The plots of solution of $b_c$* ***before*** *the Page time versus energy, $E$. Here we set $\mu = \nu = 3$, $l = 1$ and $\kappa = 2$.*

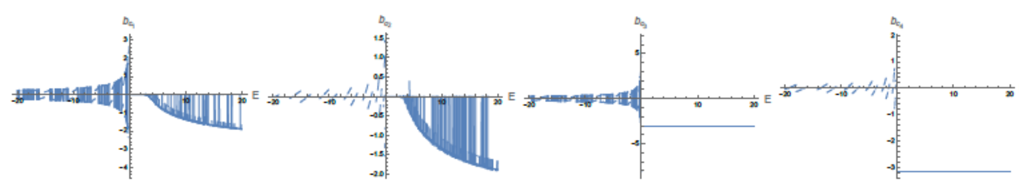
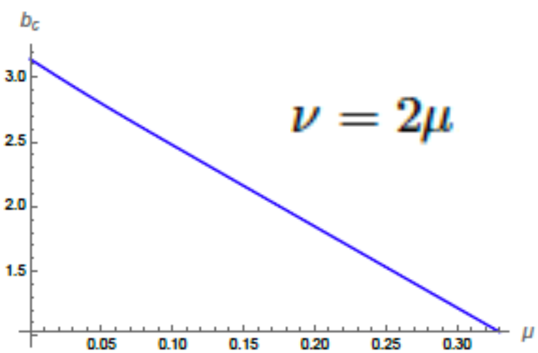
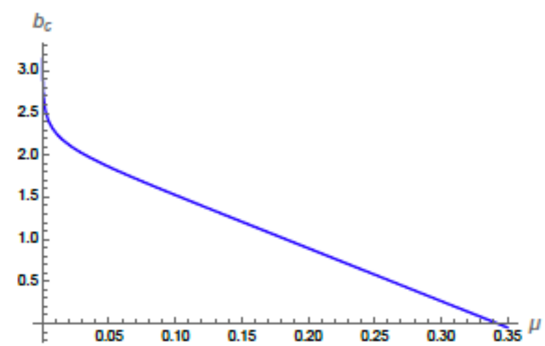


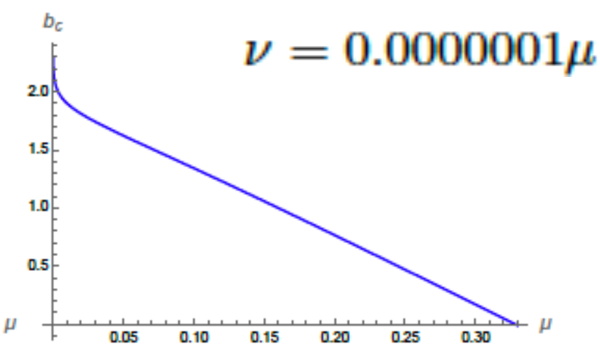
*Figure 8: The plots of solution of $b_c$* ***before*** *the Page time versus energy, $E$. Here we set $\mu = \nu = 1$, $l = 1$ and $\kappa = 3$.*

# Our main second result:

1. Before Page time, $b_c$ is much bigger than **μ**, **ν**, **μ+ν**. After the Page time $b_c$ is very small.
2. These matches with negativity, as where **μ**, **ν**, **μ/ν** are very big, $b_c$ is a linearly decreasing function.




**4. Results match with negativity of H. Shapourian, S. Liu, J. Kudler-Flam, and A. Vishwanath.**

5. For small sized subsystems, the contribution of classical correlations is significant causing the differences in the behavior.

# After the Page time

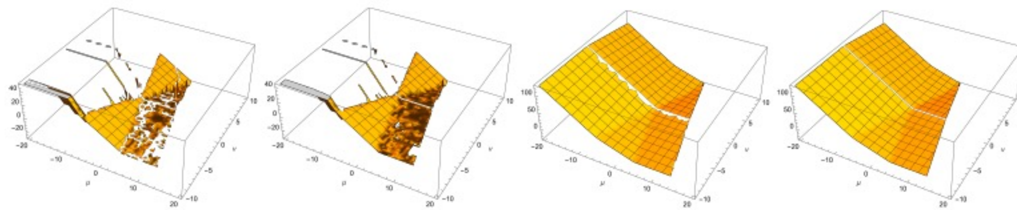
$$\sinh^2\frac{\pi}{\beta}\left(2\pi\ell\mu+2b_c\right)=\sinh\frac{\pi}{\beta}\left(2\pi\ell(2\mu+\nu)+2b_c\right)\sinh\frac{\pi}{\beta}\left(2\pi\ell\nu+2b_c\right),$$

$$b_c(1)\to\frac{\beta}{2\pi}\left(2i\pi c_1+\log\left(-\sqrt{\frac{e^{-\frac{4\pi^2 l(2\mu+\nu)}{\beta}}\left(\left(e^{\frac{4\pi^2 l\mu}{\beta}}-1\right)^2 e^{\frac{4\pi^2 l\nu}{\beta}}-\eta\right)}{2\left(e^{\frac{4\pi^2 l\nu}{\beta}}-1\right)}}\right)\right)$$

$$b_c(2)\to\frac{\beta}{4\pi}\left(4i\pi c_1+\log\left(\frac{e^{-\frac{4\pi^2 l(2\mu+\nu)}{\beta}}\left(\left(e^{\frac{4\pi^2 l\mu}{\beta}}-1\right)^2 e^{\frac{4\pi^2 l\nu}{\beta}}-\eta\right)}{2\left(e^{\frac{4\pi^2 l\nu}{\beta}}-1\right)}\right)\right)$$

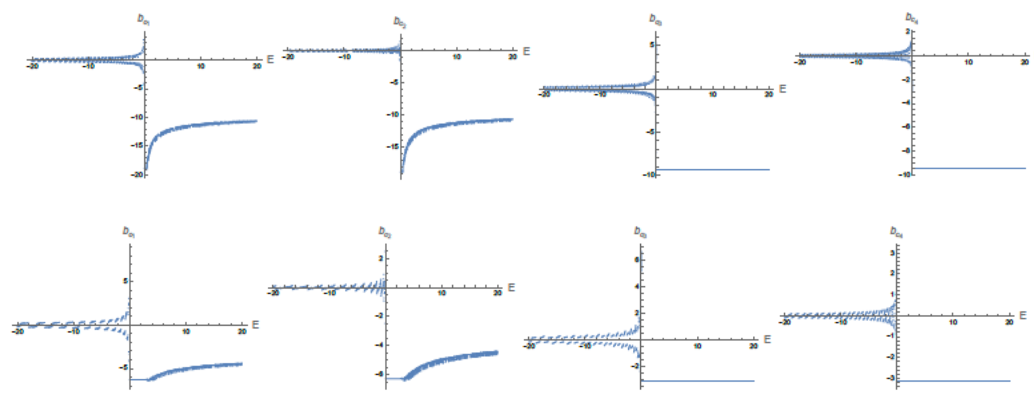
$$b_c(3)\to\frac{\beta}{2\pi}\left(2i\pi c_1+\log\left(-\sqrt{\frac{e^{-\frac{4\pi^2 l(2\mu+\nu)}{\beta}}\left(\left(e^{\frac{4\pi^2 l\mu}{\beta}}-1\right)^2 e^{\frac{4\pi^2 l\nu}{\beta}}+\eta\right)}{2\left(e^{\frac{4\pi^2 l\nu}{\beta}}-1\right)}}\right)\right)$$

$$b_c(4)\to\frac{\beta}{4\pi}\left(4i\pi c_1+\log\left(\frac{e^{-\frac{4\pi^2 l(2\mu+\nu)}{\beta}}\left(\left(e^{\frac{4\pi^2 l\mu}{\beta}}-1\right)^2 e^{\frac{4\pi^2 l\nu}{\beta}}+\eta\right)}{2\left(e^{\frac{4\pi^2 l\nu}{\beta}}-1\right)}\right)\right)$$

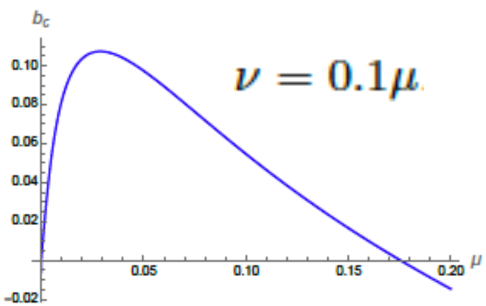


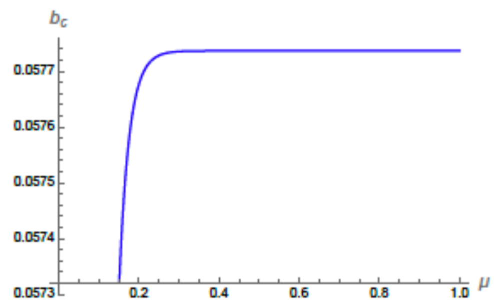
$\beta=\pi/\sqrt{\kappa E}$

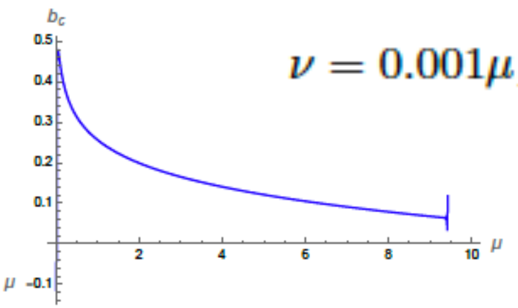
$b_c$ versus **$\mu$, $\nu$.**



# Some other results




1. Bigger size and ratio, matches with negativity.

2. The behavior of $b_c$ versus **μ**, **ν** completely changes before and after the Page time.

3. The critical bath size compared to **μ**, **ν** is very small, since most of the d.o.f of BH is already evaporated and there is less quantum correlations between **μ**, **ν**.

**Our 2nd result**

# Quantum error correction and BH Phases

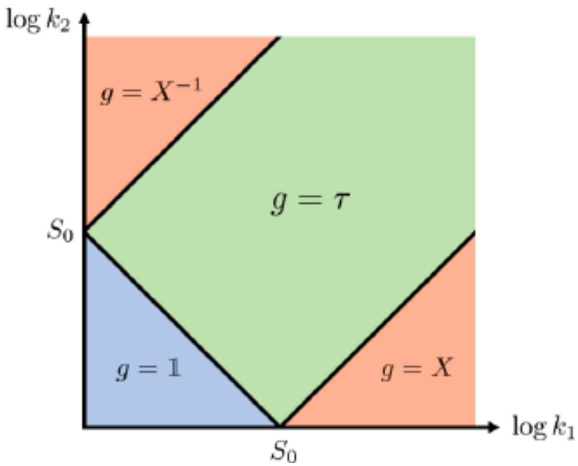
V. Balasubramanian, A. Kar, Yue, Li, O. Parrikar, arXiv:2203.01961 showed that the interior is robust against the generic low-rank operations. The noise on the density matrix of the interior of BH coming from operations on bath can be corrected. The bound on the noise would depend on the black hole entropy and the code subspace dimension.

For JT, the recovery channel satisfies: $\max_\rho ||\mathcal{R}(\mathcal{E}(\rho)) - \rho||_1 \leq \epsilon, \qquad \epsilon \sim e^{-S_0/4},$

Here t-1 is the max # of qubits in the physical Hilbert space that can tolerate error, d is dim of code space and $2^S$ is the dim of physical Hilbert space. → $2(t-1) \leq (\mathcal{S} - \log_2 d)$

# Quantum error corrections and BH Phase transitions

The connected saddle dominates the disconnected one when:




**Our main 3rd result**

from $g = I$ to $g = \tau$

$$\frac{(\frac{\ell}{k})^{n-1} + k^{n-1}}{1 + \ell^{n-1}} \approx \left(\frac{k}{\ell}\right)^{n-1} \ll e^{(1-n)(S_{BH}+\log d_i)},$$

$k = k_1 k_2 \quad C_m = \frac{1}{m+1}\binom{2m}{m}$

for even m:

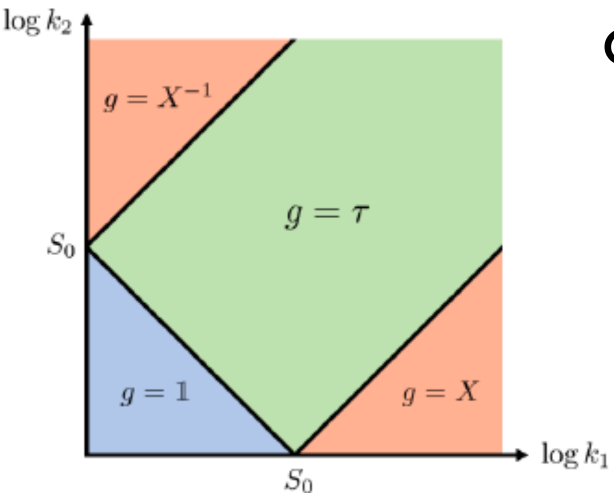
$$\frac{Z_2^m}{Z_1^{2m}} \sim e^{(1-m)S_{BH}} > \frac{d_e^{m-1}}{k^m C_m},$$

$$Z_n = e^{S_0} \int_0^\infty ds \rho(s) y(s)^n.$$

for odd m:

$$\frac{Z_2^{m-1}}{Z_1^{2m-2}} > \frac{d_e^{m-1}}{k^{m-1} C_{m-1}(2m-1)}.$$

# Quantum error corrections and BH Phase transitions




**Our main 3rd result**

from $g = X$ to $g = \tau$

for even m:

$$\frac{Z_{2m}}{Z_2^m} < \frac{k_2^{2m-2} C_m}{d_i^m d_e^m k^{m-1}},$$

for odd m:

$$\frac{Z_{2m-1}}{Z_1 Z_2^{m-1}} < \frac{(2m-1) C_{m-1}}{d_i^m d_e^m k^{m-1}},$$

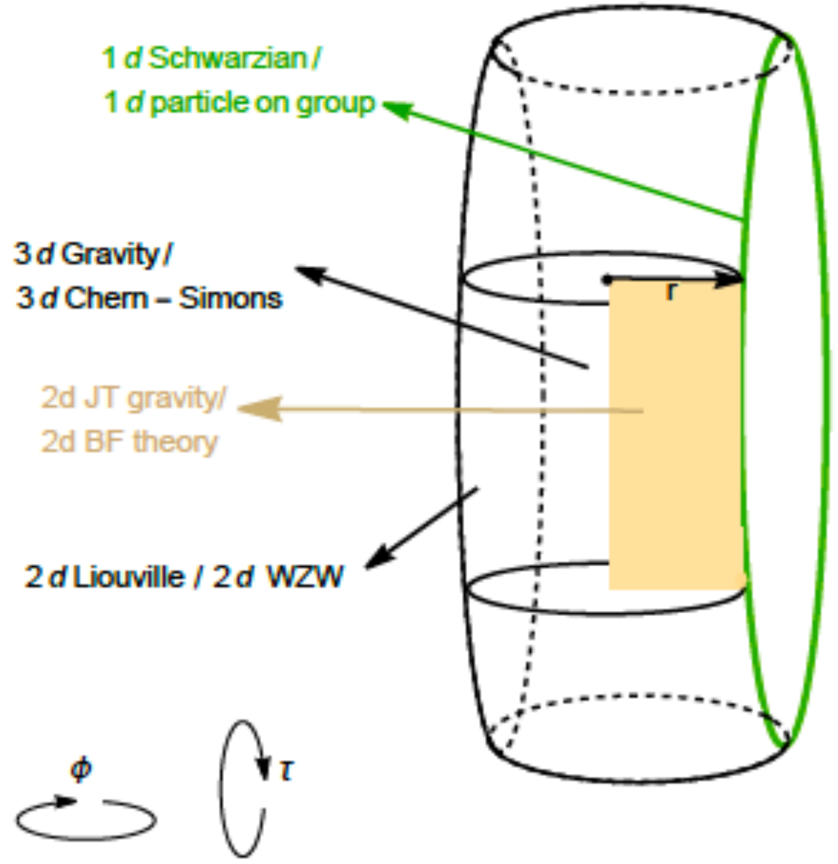
For going from $g = X^{-1}$ to $g = \tau$, one only needs to swap $k_2$ to $k_1$ in the above relations.

Note that g = I cannot reach to g = X or $g = X^{-1}$, and also g = X is further away from $g = X^{-1}$, which can also be seen from the relations between the dimensions of the code subspaces.

# Partition function of 3d versus 2d

Theories in 3d and 2d are connected through **holography** or **dim reduction**. The 3d gravity leads to 2d JT gravity by dim-reduction and to 2d Liouville theory by holography.

The 2d Liouville CFT leads to 1d Schwarzian theory by dim-reduction. The 3d Chern-Simons is related to 2d BF theory by dim-reduction and to 2d WZW model by holography. The 2d WZW, model after the dimensional reduction, leads to the model for 1d particle on group.



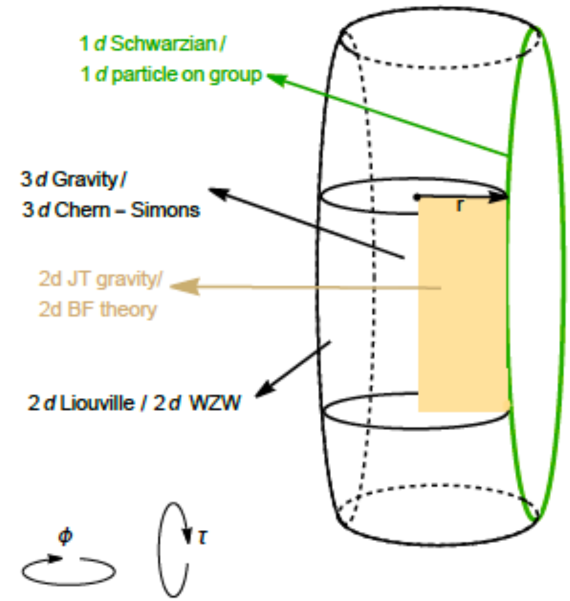
# JT vs. BF and Liouville vs. WZW

Log k which is related to entropy $S_0$ of JT is related to orthonormal states, would become Log $k_L$ in WZW.

The term $\sqrt{h}\phi K$ in 2d JT model corresponds to $\chi A_0$ in 2d BF model and the term $\sqrt{g}\phi R$ in JT corresponds to $\chi F$ term in BF.

Also, the Liouville momentum which is a continous parameter labeled by k is related to the continous irreducible representation of SL(2,R) labeled by R.

Also, the defect in the 2d JT model such as conical defects or wormholes is related to the deformation of the 1d Schwarzian theory.



# WZW vs Liouville

The two parameters of $\Phi_h$ which is the **horizon area operator** and L($\gamma$) which is the **geodesic length operator** is related to the deformation of the particle-on-a-group quantum mechanics where a "**chemical potential**" or "**magnetic monopole**" is being added.

$SL(2,\mathbb{R})$ WZW

$$S_{WZW}[g] = \frac{k_L}{8\pi}\int d\tau d\sigma \sqrt{-h}\,\mathrm{Tr}(\partial_a g^{-1}\partial^a g) + k_L \Gamma_{WZ}[g],$$
$$\Gamma_{WZ}[g] = -\frac{1}{12\pi}\int \epsilon^{abc}\,\mathrm{Tr}\,(\partial_a g\, g^{-1}\partial_b g\, g^{-1}\partial_c g\, g^{-1}),$$

For the compact manifold, $k_L \in \mathbb{Z}$, and it is related to the entropy of black hole at zero temperature, $S_0$ in JT.

**Our 4th result**

# Boundary effect (shift)

$$S = \frac{1}{2}\oint_{\partial M} dt \mathrm{Tr}(g^{-1}\partial_t g)^2.$$

$$H = H_{\mathrm{grav}} + H_{CS}$$

The boundary would only affect the Hamiltonian and causes an energy shift in the form of →

$$T_{tt} = \frac{\dot{\sigma}^2}{2},$$

The effects of this boundary term in the gauged case is like a quench in the BCFT, as it is like injections of energy.

**1st law of EE leads to**

$$T_{tt}(x_A, t) = \lim_{|x_A - x_B| \to 0} \frac{3}{\pi |x_A - x_B|^2} \cdot \Delta S_{A,B}(x_A, x_B, t).$$

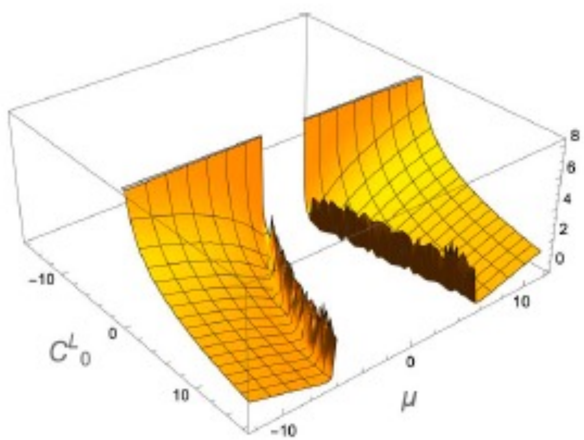
$$\Delta S = \frac{\pi}{6} \frac{\Delta \dot{\varphi}^2}{\lim_{|d\varphi| \to 0} \frac{1}{d\varphi^2}}$$

← This entropy would be created due to the injection of the above energy.

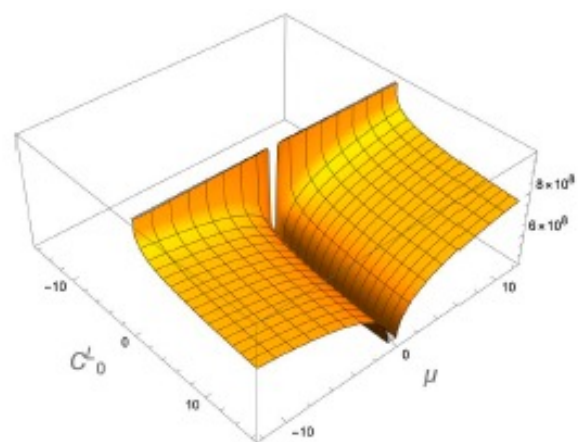
**Our 4th result**

# Liouville to JT from Island

The Liouville gravity amplitude and JT gravity partition functions are also related to each other by a double scaling limit, b → 0, where the boundary length go to infinity, $\ell \sim \ell_{JT}/(\boldsymbol{\kappa}\ b) \rightarrow +\infty$. In this limit the diagram of EE becomes smoother. By increasing the parameters of the potential **μ** and **b**, the entanglement of the island would decrease, and since **b** is in the exponent, it has a bigger effect.
The limit b → 0 is related to genus zero of Weil-Petersson volume.



Liouville



JT

**Our 4th result**

# Speed of the island

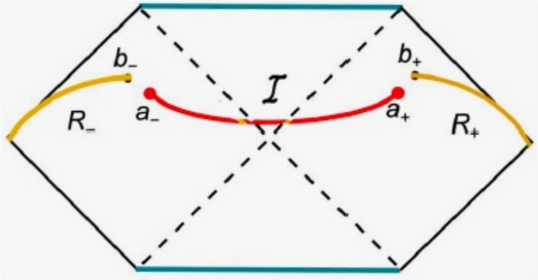
In the late time, the island appears barely outside of the horizon, while in early times of radiation, the island is inside of the black hole as it moves or expands toward the horizon gradually by the effects of the momentum of the Hawking radiations.

Understanding the movement of island inside of BH → Renormalization flow. For the metric:

$$ds^2 = e^{2A(\rho)}[-f(\rho)^2 dt^2 + d\vec{x}^2] + d\rho^2,$$

The a-function has the form

$$a_T(\rho) = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)\ell_P^{d-1}}\left[\frac{f(\rho)}{A'(\rho)}\right]^{d-1}$$




$$v_{\text{island}} \propto \frac{da_T}{d\rho} = \frac{(d-1)\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)\ell_P^{d-1}}\frac{f(\rho)^{d-2}}{A'(\rho)^d} \times [f'(\rho)A'(\rho) - f(\rho)A''(\rho)].$$

# Movement of the island

The higher genus partition functions get **oscillatory behavior** in **k** which affects island to move from inside of BH towards the horizon and slightly outside of it in the late times. The gravity makes the CFT partition function and the spectral form factor non-smooth. The speed of this movement is proportional to the level **k**.

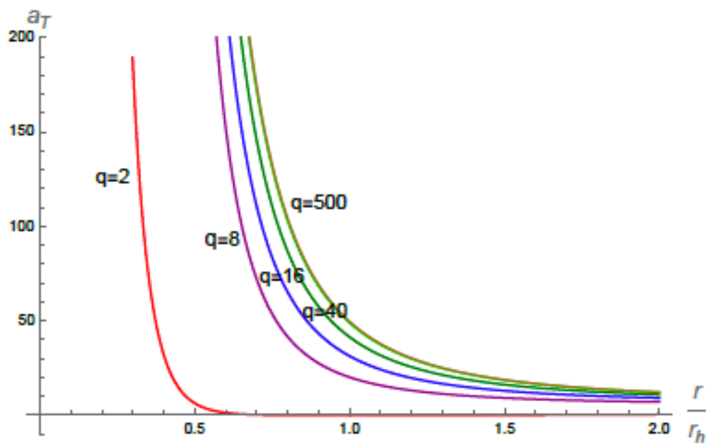
The monotonicity of mutual information between island and radiation.

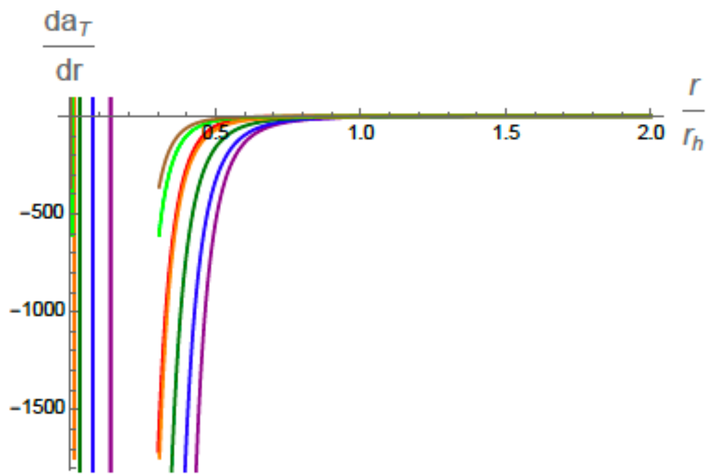
$$\frac{d}{d\lambda_+} I_{\text{mat}}(I, R) \geq 0,$$

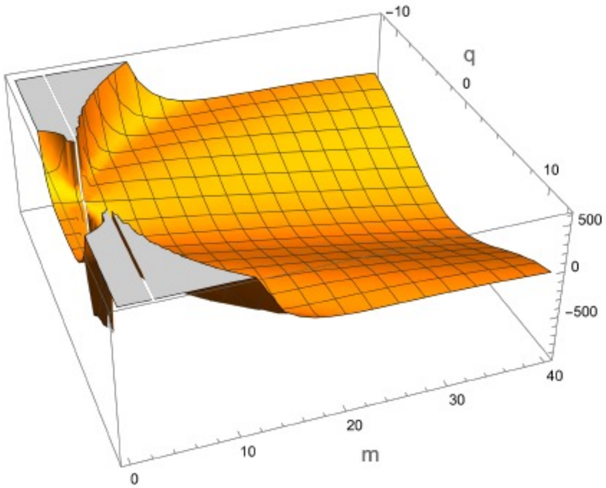
Island should move towards the horizon, closer to the radiation part to keep this inequality. So these ideas are consistent.

**Our 5th result**

# The a-function for Schwarzchild BH




The speed is higher close to singularity and decreases toward the horizon. The mass of graviton slows down the movement of the island and charge can increase it, as one would expect.

**Our 5th result**

# Summary

1. In the phase space of black hole radiation, the phase which breaks the replica symmetry should cover most of the space.

2. Tracing the mixed correlation and mixed information from 1d higher point of view in the setup of VV would give consistent results.
3. Applying noise would derive the phase transitions between the saddles of mixed radiations and consistent inequalities have been derived when moving between these saddles.
4. In the gauged formulation of 2d gravity, the boundary term only inject a shift in the phase space. Also, the Chern level $k_L$ play the role black hole entropy in JT. In addition, in the setup of 2d gravities, turning off Liouville potential would lead consistent results for island to JT.
5. Using renormalization group flow, a velocity for island can be introduced which gives consistent results with the behavior of MI between Island and black hole interior.

# Thanks for Your Attention!