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Einstein Field Equations and Black Holes

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Outline

- Einstein theories and equations of motion
- Exact solutions, classifications and techniques
- Global analysis: black rings
- Euclidean signature: gravitational instantons.
- Some general properties by energy conditions
- Regular black holes

We consider symmetric metric $g_{\mu\nu} = g_{\nu\mu}$, and are not going to deal with torsion.

Einstein Theory of Gravity(1915)

$$G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

where $G_{\mu\nu}$ is called Einstein tensor. Specifically, for the metric $ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu , \quad x^\mu = \{t, x, y, z\}$

$$\begin{aligned}
g &\equiv \det(g_{\mu\nu}) = \tfrac{1}{24} \varepsilon^{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma\delta} g_{\mu\alpha} g_{\nu\beta} g_{\rho\gamma} g_{\sigma\delta} , \\
g^{\mu\alpha} &= \frac{1}{6g} \varepsilon^{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma\delta} g_{\nu\beta} g_{\rho\gamma} g_{\sigma\delta} , \qquad g_{\mu\lambda} g^{\nu\lambda} = \delta^\nu_\mu , \\
\Gamma^\rho{}_{\mu\nu} &= \tfrac{1}{2} g^{\rho\lambda} (\partial_\nu g_{\lambda\mu} + \partial_\mu g_{\lambda\nu} - \partial_\lambda g_{\mu\nu}) , \\
R^\rho{}_{\sigma\mu\nu} &= \partial_\mu \Gamma^\rho{}_{\nu\sigma} - \partial_\nu \Gamma^\rho{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\lambda} \Gamma^\lambda{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\lambda} \Gamma^\lambda{}_{\mu\sigma} , \\
R_{\mu\nu} &= R^\rho_{\mu\rho\nu} , \qquad R = R^\mu{}_\mu .
\end{aligned}$$

$\varepsilon^{\mu\rho\rho\sigma}$ is the totally-antisymmetric tensor density with $\varepsilon^{0123} = 1$ and $\epsilon^{\mu\nu\rho\sigma} = \varepsilon^{\mu\rho\rho\sigma} / \sqrt{|g|}$ is a tensor.

(Note that Riemann tensor depends on Γ, but not explicitly on $g_{\mu\nu}$.) Torsion free: $\Gamma^\rho{}_{\mu\nu} = \Gamma^\rho{}_{\nu\mu}$. Also metricity: $\nabla_\mu g_{\nu\rho} = 0$.

Focus on four dimensions

Although we shall discuss Einstein gravity in general dimensions, we shall focus on four dimensions. The theory seems to be just about right only in four dimensions

- # of Riemann tensor components $\frac{1}{12}D^2(D^2-1)$
- # of Ricci tensor components $\frac{1}{2}D(D+1)$
- # of Weyl tensor components $\frac{1}{12}D^2(D^2-1)-\frac{1}{2}D(D+1)$.

In higher dimensions, the Riemann tensor components become far greater than the Ricci tensor components. In lower dimensions, the theory becomes two degenerate.

Dim	Riemann	Ricci	Weyl
1	0	0	0
2	1	1	0
3	6	6	0
4	20	10	10
5	50	15	35
6	105	21	84

Vacuum equations

In the vacuum, (assuming there is no vacuum energy,) we have $T_{\mu\nu} = 0$, the equations become

$$G_{\mu\nu} = 0\,, \qquad i.e. \qquad R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0\,.$$

Taking the trace, for $D \neq 2$, we have $R = 0$ and hence

$$R_{\mu\nu} = 0\,.$$

A vacuum solution is not the same as the vacuum, since the source can be localized.

e.g. The spacetime of our Sun outside is the Kerr metric, which is a vacuum solution to Einstein's field equation.

Matter vs. Spacetime

We typically take the view that Einstein's equation $G_{\mu\nu} = T_{\mu\nu}$ as how spacetime curves under a certain matter energy-momentum tensor.

But this can be reversed as $T_{\mu\nu} = G_{\mu\nu}$, in that we can write any spacetime metric, and there must be a certain $T^{\mu\nu}$.

However, we have reliable theories of matter fields, which give constraints on the allowed $T^{\mu\nu}$.

These constraints can be summarized abstractly as a set of energy conditions on $T^{\mu\nu}$. This leads to some general properties on the allowed spacetime.

We shall come back to this later.

Einstein-Hilbert action

Not all the equations of motion can be derived from an action, based on the Hamilton's action principle, but Einstein's can!

$$I = \frac{1}{16\pi}\int d^4x \mathcal{L}\,, \qquad \mathcal{L} = \sqrt{-g}R\,.$$

(assumed Lorentzian signature here.) In Einstein's theory of gravity, $g_{\mu\nu}$ or $g^{\mu\nu}$ are treated as fundamental fields. Einstein equation is $E_{\mu\nu} = 0$ where $\delta\mathcal{L} = \sqrt{-g}E_{\mu\nu}\delta g^{\mu\nu} +$ total derivatives.

Some useful Lemma

$$\begin{aligned}
&\delta g = g g^{\mu\nu}\delta g_{\nu\mu} = -g g_{\mu\nu}\delta g^{\nu\mu}\,, \quad \delta\sqrt{-g} = -\tfrac{1}{2}\sqrt{-g} g_{\mu\nu}\delta g^{\nu\mu}\,,\\
&\delta\Gamma^\rho{}_{\mu\nu} = \tfrac{1}{2}g^{\rho\lambda}(\nabla_\nu\delta g_{\lambda\mu} + \nabla_\mu\delta g_{\lambda\nu} - \nabla_\lambda\delta g_{\mu\nu})\,,\\
&\delta R^\rho{}_{\sigma\mu\nu} = \nabla_\mu(\delta\Gamma^\rho{}_{\nu\sigma}) - \nabla_\nu(\delta\Gamma^\rho{}_{\mu\sigma})\,,\\
&\delta R_{\mu\nu} = \delta R^\rho{}_{\mu\rho\nu} = \nabla_\rho(\delta\Gamma^\rho{}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^\rho{}_{\rho\mu})\,.
\end{aligned}$$

Note that although $\Gamma^\rho{}_{\mu\nu}$ is not a tensor, but $\delta\Gamma^\rho{}_{\mu\nu}$ is.

Since $R = g^{\mu\nu} R_{\mu\nu}$, we have

$$\begin{aligned} \delta R &= R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu} \\ &= R_{\mu\nu}\delta g^{\mu\nu} + \nabla_\sigma \underbrace{(g^{\mu\nu}\delta\Gamma^\sigma{}_{\nu\mu} - g^{\mu\sigma}\delta\Gamma^\rho{}_{\rho\mu})}_{\equiv X^\sigma} \\ &= R_{\mu\nu}\delta g^{\mu\nu} + \nabla_\sigma X^\sigma \\ &= R_{\mu\nu}\delta g^{\mu\nu} + \frac{1}{\sqrt{-g}}\partial_\sigma(\sqrt{-g}X^\sigma) \, . \end{aligned}$$

Thus

$$\delta(\sqrt{-g}R) = \sqrt{-g}\underbrace{(R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu})}_{G_{\mu\nu}}\delta g^{\mu\nu} + \underbrace{\sqrt{-g}\nabla_\sigma X^\sigma}_{\text{total derivative}}$$

(Gibbons-Hawking boundary term is needed for applying the boundary condition in the Hamilton's action principle. We shall be sloppy here.)

Lagrangian and conserved currents

It took awhile for Einstein to realize that it should be $G_{\mu\nu}$ rather than $R_{\mu\nu}$ that equals to $T_{\mu\nu}$, owing to the following **identity**

$$\nabla^\mu R_{\mu\nu} = \frac{1}{2}\nabla_\nu R\,, \qquad \longrightarrow \qquad \nabla^\mu G_{\mu\nu} = 0\,.$$

This ensures that the conservation of energy-momentum tensor $\nabla^\mu T_{\mu\nu} = 0$ is consistent with geometrical properties.

Such identity requires fluent knowledge on geometries and not easy to generalize to more complicated theories, e.g., non-mimimal coupling, or higher-derivative gravities.

Lagrangian formulation, on the other hand, makes the statement trivial, by the virtual of Noether theorem.

(Exercise: demonstrate that in a general-covariant theory, the quantity $E_{\mu\nu} \equiv \frac{\delta\mathcal{L}}{\delta g^{\mu\nu}}$ must be conserved, i.e. $\nabla^\mu E_{\mu\nu} = 0$.)

A short list of theories

Einstein-Maxwell theory

$$\mathcal{L} = \sqrt{-g}(R - \frac{1}{4}F^2)\,, \qquad F = dA\,, \qquad A = A_\mu dx^\mu\,.$$

In other words, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $F^2 = F_{\mu\nu}F^{\mu\nu}$. Bianchi identity of the Maxwell fields $dA = 0$ is automatically satisfied, i.e. $\partial_{[\mu}F_{\nu\rho]} = 0$.

Equations of motion

$$\begin{aligned} \delta A_\mu: &\quad \nabla_\mu F^{\mu\nu} = 0\,, \\ \delta g^{\mu\nu}: &\quad G_{\mu\nu} = \tfrac{1}{2}(F^2_{\mu\nu} - \tfrac{1}{4}g_{\mu\nu}F^2)\,, \end{aligned}$$

where $F^2_{\mu\nu} \equiv F_{\mu\rho}F_\nu{}^\rho$. Note that in $D = 4$, the energy-momentum tensor is traceless, indicating that the Maxwell field is conformal.

Einstein-Maxwell theory with Λ

$$\mathcal{L} = \sqrt{-g}(R - 2\Lambda - \frac{1}{4}F^2)\,.$$

Equations of motion

$$\begin{aligned} \delta A: &\quad \nabla_\mu F^{\mu\nu} = 0\,, \\ \delta g^{\mu\nu}: &\quad G_{\mu\nu} + \Lambda g_{\mu\nu} = \tfrac{1}{2}(F^2_{\mu\nu} - \tfrac{1}{4} g_{\mu\nu} F^2)\,. \end{aligned}$$

Einstein-Proca theory with Λ

$$\mathcal{L} = \sqrt{-g}(R - 2\Lambda - \frac{1}{4}F^2 - \tfrac{1}{2}m^2 A^2)\,, \qquad A^2 = A^\mu A_\mu$$

Equations of motion

$$\begin{aligned} \delta A: &\quad \nabla_\mu F^{\mu\nu} = m^2 A^\nu\,, \\ \delta g^{\mu\nu}: &\quad G_{\mu\nu} + \Lambda g_{\mu\nu} = \tfrac{1}{2}(F^2_{\mu\nu} - \tfrac{1}{4}g_{\mu\nu}F^2) \\ &\qquad\qquad\qquad\qquad + \tfrac{1}{2}m^2(A_\mu A_\nu - \tfrac{1}{2}A^2 g_{\mu\nu})\,. \end{aligned}$$

Einstein-Scalar theory

Minimally-coupled scalar

$$\mathcal{L} = \sqrt{-g}(R - \tfrac{1}{2}(\partial\phi)^2 - V(\phi))\,.$$

Equations of motion

$$\delta\phi: \qquad \Box\phi = \frac{\partial V}{\partial\phi}\,,$$

$$\delta g^{\mu\nu}: \qquad G_{\mu\nu} = \tfrac{1}{2}(\partial_\mu\phi\partial_\nu\phi - \tfrac{1}{2}(\partial\phi)^2 g_{\mu\nu}) - \tfrac{1}{2}V\, g_{\mu\nu}\,.$$

Einstein-Maxwell-Dilaton theory

In string theory, the following theory is a common occurrence

$$\mathcal{L} = \sqrt{-g}\Big(R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{4}e^{a\phi}F^2 - V(\phi)\Big)\,.$$

Equations of motion

$$\begin{aligned}
\delta\phi: &\qquad \Box\phi = \frac{a}{4}e^{a\phi}F^2 + \frac{\partial V}{\partial\phi}\,,\\
\delta A_\mu: &\qquad \nabla_\mu(e^{a\phi}F^{\mu\nu}) = 0\,,\\
\delta g^{\mu\nu}: &\qquad G_{\mu\nu} = \tfrac{1}{2}(\partial_\mu\phi\partial_\nu\phi - \tfrac{1}{2}(\partial\phi)^2 g_{\mu\nu}) - \tfrac{1}{2}V g_{\mu\nu}\\
&\qquad\qquad +\tfrac{1}{2}e^{a\phi}(F^2_{\mu\nu} - \tfrac{1}{4}g_{\mu\nu}F^2)\,.
\end{aligned}$$

Note that all the vector $A_{(1)} = A_\mu dx^\mu$ can be generalized to an n-form $A_{(n)} = A_{\mu_1\cdots A_{\mu_n}} dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_n}$.

hep-th/9412184, hep-th/9508042, [1306.2386]

Einstein-Yang-Mills theory

$$\mathcal{L} = \sqrt{-g}\Big(R - 2\Lambda - \frac{1}{2g_s^2} F^a_{\mu\nu} F^{a\mu\nu}\Big)\,.$$

where the $SU(2)$ Yang-Mills field strength is defined by:

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc} A^b_\mu A^c_\nu\,.$$

equations of motion:

$$\nabla_\mu F^{a\mu\nu} + \epsilon^{abc} A^b_\mu F^{c\mu\nu} = 0\,,$$
$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{1}{g_s^2}(g^{\rho\sigma} F^a_{\mu\rho} F^a_{\nu\sigma} - \frac{1}{4} F^2 g_{\mu\nu})\,.$$

(Here a, b, c etc are Yang-Mills group indices.)

Non-Minimally coupled scalars

$$\mathcal{L} = \sqrt{-g}\Big(\kappa_0 R - \tfrac{1}{2}\xi\phi^2 R - \tfrac{1}{2}(\partial\phi)^2 - V(\phi)\Big)\,.$$

κ_0, ξ are constants. equations of motion:

$$\begin{aligned}
\Box\phi &= \xi\phi R + \frac{dV}{d\phi}\,, \\
\kappa_0 G_{\mu\nu} &= \tfrac{1}{2}\partial_\mu\phi\partial_\nu\phi - \tfrac{1}{2}g_{\mu\nu}\Big(\tfrac{1}{2}(\partial\phi)^2 + V(\phi)\Big) \\
&\quad + \tfrac{1}{2}\xi(\phi^2 G_{\mu\nu} + g_{\mu\nu}\Box\phi^2 - \nabla_\mu\nabla_\nu\phi^2)\,.
\end{aligned}$$

Higher-derivative gravities

$$\mathcal{L} = \sqrt{-g}\Big((R-2\Lambda_0)+\alpha R^2+\beta\, R_{\mu\nu}R^{\mu\nu} \\ +\gamma\left(R^2-4R_{\mu\nu}R^{\mu\nu}+R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right)\Big)\,.$$

equations of motion

$$\begin{aligned} 0 \;=\; & R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R+\Lambda_0 g_{\mu\nu}+2\alpha R(R_{\mu\nu}-\tfrac{1}{4}Rg_{\mu\nu}) \\ & +(2\alpha+\beta)(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu)R \\ & +\beta\Box(R_{\mu\nu}-\tfrac{1}{2}Rg_{\mu\nu})+2\beta(R_{\mu\sigma\nu\rho}-\tfrac{1}{4}g_{\mu\nu}R_{\sigma\rho})R^{\sigma\rho} \\ & +2\gamma\Big(RR_{\mu\nu}-2R_{\mu\sigma\nu\rho}R^{\sigma\rho}+R_{\mu\sigma\rho\lambda}R_\nu{}^{\sigma\rho\lambda}-R_{\mu\rho}R_\nu{}^\rho\Big) \\ & -\tfrac{1}{2}\gamma g_{\mu\nu}\Big(R^2-4R_{\mu\nu}R^{\mu\nu}+R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\Big)\,. \end{aligned}$$

In particular, $\alpha=0=\beta$ gives the Einstein-Gauss-Bonnet theory.

[1101.1971,1101.4009,1610.08519]

With non-minmially-coupled derivative matter

$$I = \frac{1}{16\pi}\int d^n x\sqrt{-g}\,L\,, \qquad L = \kappa(R-2\Lambda) - \tfrac{1}{2}(\alpha g_{\mu\nu} - \gamma G_{\mu\nu})\partial^\mu\chi\,\partial^\nu\chi\,,$$

equations of motion

$$\begin{aligned}
0 &= \kappa(G_{\mu\nu}+\Lambda g_{\mu\nu}) - \tfrac{1}{2}\alpha\Big(\partial_\mu\chi\partial_\nu\chi - \tfrac{1}{2}g_{\mu\nu}(\partial\chi)^2\Big)\\
&\quad -\tfrac{1}{2}\gamma\Big(\tfrac{1}{2}\partial_\mu\chi\partial_\nu\chi R - 2\partial_\rho\chi\,\partial_{(\mu}\chi\,R_{\nu)}{}^\rho\\
&\quad -\partial_\rho\chi\partial_\sigma\chi\,R_\mu{}^\rho{}_\nu{}^\sigma - (\nabla_\mu\nabla^\rho\chi)(\nabla_\nu\nabla_\rho\chi)\\
&\quad +(\nabla_\mu\nabla_\nu\chi)\Box\chi + \tfrac{1}{2}G_{\mu\nu}(\partial\chi)^2\\
&\quad -g_{\mu\nu}\Big[-\tfrac{1}{2}(\nabla^\rho\nabla^\sigma\chi)(\nabla_\rho\nabla_\sigma\chi) + \tfrac{1}{2}(\Box\chi)^2 - \partial_\rho\chi\partial_\sigma\chi\,R^{\rho\sigma}\Big]\Big)\,,\\
0 &= \nabla_\mu\Big((\alpha g^{\mu\nu} - \gamma G^{\mu\nu})\nabla_\nu\chi\Big)\,.
\end{aligned}$$

Horndeski gravity

What about fermions

Anti-commuting fermion fields are described not by complex numbers, but rather by the Grassmannian numbers.

At the linear level, such as Dirac equations, the distinction is irrelevant.

However, the energy-momentum tensor is bilinear in fermionic fields, and they cannot modify the curvature with real numbers at the classical level.

One might consider semi-classical consideration, namely

$$G_{\mu\nu} = \langle T_{\mu\nu} \rangle \, .$$

But this is not quite what Einstein wrote. (Personally, I do not quite know what it means.)

Commuting fermions are fine; however, do we have any such fields in real world?

Now solving equations

Einstein's field equation is highly nonlinear. Even in four dimensions, it is a set of nonlinear second-order differential equation involving six functions ($\frac{1}{2}(4\times5)-4=6$) of four variables (t,x,y,z).

At the first sight, it is hard to imagine to construct anything nontrivial.

Trivial solution: the Minkowski metric: $ds^2=-dt^2+dx^2+dy^2+dz^2$, i.e $g_{\mu\nu}$ are all constants, and hence the connections $\Gamma^\rho{}_{\mu\nu}$, Riemann tensor $R^\mu{}_{\nu\rho\sigma}$, Ricci tensor $R_{\mu\nu}$ and Ricci scalar R all vanish。Einstein vacuum equation is automatically satisfied。

Note that if we write Minkowski space in spherically-symmetric form, namely $ds^2=-dt^2+dr^2+r^2d\Omega^2$, then all curvatures still vanishes, but the connection Γ does not.

Linearized gravity and graviton

The general equations are hard to obtain, we can consider linearized equations of motion. Consider vacuum Einstein equation with a cosmological constant

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0\,.$$

The vacuum solution is (A)dS/Mink with

$$\bar{R}_{\mu\nu\rho\sigma} = \tilde{\Lambda}(\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma} - \bar{g}_{\mu\sigma}\bar{g}_{\rho\nu})\,.$$

(Question: How $\tilde{\Lambda}$ is related to Λ?) Perform a small perturbation

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}\,, \qquad \Longrightarrow \qquad g^{\mu\nu} = \bar{g}^{\mu\nu} - h^{\mu\nu} + \mathcal{O}(h^2)\,.$$

Thus we can lower and raise the indices by $\bar{g}_{\mu\nu}$ and $\bar{g}^{\mu\nu}$ on the linearized h terms.

The linearized equation becomes

$$\mathcal{G}^L_{\mu\nu} = 0\,,$$

The linearized Einstein tensor around the (A)dS vacuum is given by

$$\begin{aligned}
\mathcal{G}^L_{\mu\nu} &= R^L_{\mu\nu} - \tfrac{1}{2}\bar{g}_{\mu\nu}R^L - \frac{2\Lambda}{D-2}h_{\mu\nu}\,, \\
R^L_{\mu\nu} &= \frac{1}{2}\Big(\bar{\nabla}^\sigma\bar{\nabla}_\mu h_{\nu\sigma} + \bar{\nabla}^\sigma\bar{\nabla}_\nu h_{\mu\sigma} - \bar{\Box}h_{\mu\nu} - \bar{\nabla}_\mu\bar{\nabla}_\nu h\Big)\,, \\
R^L &= -\bar{\Box}h + \bar{\nabla}^\sigma\bar{\nabla}^\mu h_{\mu\sigma} - \frac{2\Lambda}{D-2}h\,.
\end{aligned}$$

Taking the trace of $\mathcal{G}^L_{\mu\nu} = 0$ yields $R^L = 0$. Making a gauge choice

$$\bar{\nabla}^\mu h_{\mu\nu} = \bar{\nabla}_\nu h\,.$$

It follows that

$$R^L = -\frac{2\Lambda}{D-2}h\,.$$

Thus the physical modes is traceless, $h = 0$ and hence also transverse $\bar{\nabla}^\mu h_{\mu\nu} = 0$, which leads to $(\Box - 2/3\Lambda)h_{\mu\nu} = 0$. [1101.1971]

This leads to linearized graviton or gravitational wave with 2 degrees of freedom in four dimensions.

$$\text{useful formula}: \qquad [\nabla_\mu, \nabla_\nu]V^\rho{}_\sigma = R^\rho{}_{\lambda\mu\nu}V^\lambda{}_\sigma - R^\lambda{}_{\sigma\mu\nu}V^\rho{}_\lambda\,.$$

But we are interested in exact, not approximate solutions!

Classification: Cohomogeneity

In this lecture, we shall try to classify solutions by their cohomogeneity number.

- cohomogeneity-0: homogeneous space or spacetime, e.g. Minkowski vacua In a homoengeous space, any point can be reached by a group transitive action from any other point. Alternatively, there must exist a vielbein base such that all curvature tensors are independent of coordinates.

- cohomogeneity-1: homogeneous after one coordinate is set to be a constant, e.g. Schwarzschild black hole, FLRW models.

- cohomogeneity-n: homogeneous after n-coordinates are set to be constants, e.g. Rotating black holes, Gibbons-Hawking multiple instantons.

Vacua: cohomogeneity-0 or homogeneous

Maximally symmetric

- Minkowski: $R_{\mu\nu\rho\sigma} = 0$
- (A)dS: $R_{\mu\nu\rho\sigma} = \tilde{\Lambda}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})$.

Non-Maximally-symmetric

- Products of Maximally-symmetric spaces, e.g. $\mathrm{AdS}_5 \times S^5$
- Lifshitz spacetimes and their generalizations
- Schrödinger spacetimes and their generalization
- Squashed spheres and their generaizations.
- ...?

Lifshitz spacetimes

$$ds^2 = \ell^2\Big(-r^{2z}dt^2 + \frac{dr^2}{r^2} + r^2 dx^i dx^i\Big)\,,$$

The metric is homogeneous and invariant under the scaling

$$t \to \lambda^z t\,, \qquad x^i \to \lambda x^i\,, \qquad r \to \lambda^{-1} r\,.$$

i.e. scales differently in temporal and spatial directions. (c.f. Schrödinger equation for a free particle.)

The easiest to construct is by the Einstein-Proca theory, with $A = qr^z dt$, $\Lambda = \frac{1}{2}(D-1)(D-2)g^2$ and

$$\ell^2 = \frac{(D-2)z}{m^2}\,, \qquad m^2 = \frac{(D-1)(D-2)^2 g^2 z}{z^2 + (D-3)z + (D-2)^2}\,,$$
$$q^2 = \frac{2(z-1)(z^2 + (D-3)z + (D-2)^2)}{(D-1)(D-2)g^2 z}\,.$$

[1310.8348]

Schrödinger solutions

The metric

$$ds^2 = \ell^2\Big(- r^{2z}dt^2 + \frac{dr^2}{r^2} + r^2(-2dxdt + dy^i dy^i)\Big)\,.$$

In Einstein-Proca theory, the ansatz is $A = qr^z dt$. The solution is [1310.8348]

$$\ell = g^{-1}\,,\qquad m^2 = z(z+D-3)g^2\,,\qquad q^2 = \frac{2(z-1)}{zg^2}\,.$$

Both Lifshitz and Schrödinger solutions belong to the following general homogeneous metrics

$$ds^2 = d\rho^2 + \sum_{\mu,\nu} e^{z_{\mu\nu}\rho} dx^\mu dx^\nu c_{\mu\nu}\,,$$

where $c_{\mu\nu}, z_{\mu\nu}$ are constants. [1303.5781]

The self-interacting of Yang-Mills provides a natural source of Lifshitz and Schrdoinger solutions: [1501.01727, 1501.05318]

Cohomogeneity-one solutions

- Most of cosmological models, e.g., FLRW, Bianchi-IX, etc, $ds^2 = -dt^2 + a(t)^2 dx^i dx^i$.
- Domain Walls, e.g. $ds^2 = d\rho^2 + a(\rho)^2 dx^\mu dx^\nu \eta_{\mu\nu}$.
- Spherically-symmetric and static, e.g., Schwarzschild black hole, wormholes, solitons, etc.
- p-branes, S-branes, etc., (subjects in string theories.)
- pp-waves as infinitely-boosted black holes.
- ...

Spherically-symmetric and static

Schwarzschild (1916):

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega_2^2$$

$$f = 1 - \frac{2m}{r}, \qquad d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2$$

This is a vacuum solution $G_{\mu\nu} = 0$.

Asymptotic ($r \to \infty$) flat (Minkowski): $f \to 1$ and $ds^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2$

In high school, one is very much fascinated by the similarity between Newton's gravity $F \sim -m_1 m_2/r^2$ and Coulomb force $F \sim q_1 q_2/r^2$.

The Schwarzschild metric makes the similarity an illusion.

Or does it?

Kerr-Schild Form

An interesting property:

$$-f dt^2 + \frac{dr^2}{f} = -f(dt^2 - \frac{dr^2}{f^2}) = -f(dt + \frac{dr}{f})(dt - \frac{dr}{f})$$

Define $du = dt + \frac{dr}{f}$, we have

$$\begin{aligned} ds^2 &= 2dudr - du^2 + r^2 d\Omega_2^2 + \frac{2m}{r} du^2 \\ &= -d\tilde{t}^2 + dr^2 + r^2 d\Omega_2^2 + \frac{2m}{r}(d\tilde{t} + dr)^2 \,, \end{aligned}$$

where $u \to \tilde{t} + r$. In other words, a black hole is a linear perturbation of the Minkowski spacetime.

For $k = d\tilde{t} + dr$, then we have $k^\mu k_\mu = 0$, i.e., it is a null vector.

The quantity $2m/r$ is analogous to the electric potential.

A relation between electromagnetic force and gravity force?

Double copy formalism

Consider the metric

$$g_{\mu\nu} = \eta_{\mu\nu} + \phi k_\mu k_\nu \,,$$

where k is a null geodesic congruence $k_\mu k^\mu = 0 = k^\mu \nabla_\mu k_\nu$ and ϕ is a scalar.

Define a Maxwell field

$$A_\mu = \phi k_\mu \,, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,.$$

Then the Einstein's field equation

$$R_{\mu\nu} = T_{\mu\nu} - \frac{1}{D-2} g_{\mu\nu} T \,,$$

reduces to

$$\partial_\nu F^{\mu\nu} = J^\mu \,, \qquad J^\mu = \pm\Big(\frac{1}{D-2}\delta^\mu_0 T - T^\mu{}_0\Big) \,.$$

Double Kerr-Schild form [hep-th/0405061]

$$ds^2 = d\bar{s}^2 + U(k_M dx^M)^2 + V(l_M dx^M)^2 \,,$$
$$k^M k_M = l^M l_M = k^M l_M = 0 \,, \qquad k^M \bar{\nabla}_M k_N = l^M \nabla_M l_N = 0 \,.$$

Schwarzschild-(A)dS

Schwarzschild black hole is asymptotic flat, but our universe has a cosmological constant.

Maximal symmetric spacetime in Einstein theory is Minkowski

Maximal symmetric spacetime in Einstein theory with a cosmological constant is (Anti-)de Sitter or (A)dS.

Schwarzschild-(A)dS:

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega_2^2$$

$$f = 1 - \tfrac{1}{3}\Lambda r^2 - \frac{2m}{r}$$

The cosmological constant in our universe is too small to be testable within our solar system.

(A)dS black hole with different topologies?

Whilst asymptotic flat black hole can have only one topologies in four dimensions, and limited topologies in higher dimensions, (back to this point later.) In asymptotic AdS spaces, you can also have tori or hyperbolic horizons. If one has already construct the spherically solutions, it is rather straightforward to take a limit to obtain general solutions. Start with

$$ds^2 = -hdt^2 + \frac{dr^2}{f} + r^2 d\Omega^2\,,$$

We let $r = \tilde{r}/\sqrt{k}$ and hence $dr^2 = d\tilde{r}^2/k$. Define $d\Omega^2 = kd\tilde{\Omega}^2$. Thus the metric becomes

$$ds^2 = -hdt^2 + \frac{d\tilde{r}^2}{kf} + \tilde{r}^2 d\tilde{\Omega}^2\,,$$

where the Ricci-tensor for the $d\tilde{\Omega}^2$ is $\tilde{R}_{ij} \sim k\tilde{g}_{ij}$.

You cannot take $k=0$ or $k=-1$ whilst still being a black hole for asymptotically-flat black holes. For (A)dS black holes, however, $h \sim r^2 = \tilde{r}^2/k \sim f$. Thus define $\tilde{h} = kh$ and $\tilde{f} = kf$, $\tilde{t} = t/\sqrt{k}$, we have

$$ds^2 = -\tilde{h} d\tilde{t}^2 + \frac{d\tilde{r}^2}{\tilde{f}} + \tilde{r}^2 d\tilde{\Omega}_k^2 \,.$$

Dropping the tilde in the solution, we get the metric is general topologies with

$$h \sim f = g^2 r^2 + k + \cdots \,.$$

This procedure can be generally done, without needing to solve any equations of motion.

If the cosmological vanishes, i.e. $g^2 = 0$, k has to be 1 or positive for the solution to have the well-defined asymptotics.

It is a misnomer to say these black holes have different topologies, since traditionary, when we say black holes can only have the sphere topology, we mean that their asymptotic infinity remains either Minkowskian or global AdS. If one is allowed to change the asymptotic, then even the Schwarzschild black hole can have different topology since we can replace the $d\Omega_2^2$ metric by any Einstein metric with positive cosmological constant and the solution still satisfies Einstein's equation.

Cohomogeneity-2 solutions

- Rotating black holes in $D = 4$
- (Spherically-symmetric) black hole formation
- A periodic array of Schwarzschild black holes (axial symmetric). [hep-th/9609126]
- C-metrics, black ring, etc.
- ...?

Rotation on spacetime

From Newtonian gravity, the gravitational field created by the Sun (assuming its spherically symmetric) is independent of its rotation.

Einstein theory predicts that the rotation of the matter can drag the spacetime around it.

This is an important difference between the two theories, which provides an experimental test of the two theories.

Rotating Black Hole: Kerr Solution

Kerr metric (1963): $G_{\mu\nu} = 0$

$$
\begin{aligned}
ds^2 &= \rho^2\Big(\frac{dr^2}{\Delta_r} + d\theta^2\Big) + \frac{\sin^2\theta\,\Delta_\theta}{\rho^2}\Big(a dt - (r^2+a^2)d\phi\Big)^2 \\
&\quad -\frac{\Delta_r}{\rho^2}\Big(dt - a\sin^2\theta\, d\phi\Big)^2,
\end{aligned}
$$

$$
\rho^2 = r^2 + a^2\cos^2\theta\,, \qquad \Delta_r = r^2 + a^2 - 2mr
$$

- Mass: $M = m$
- Angular momentum: $J = ma$
- $J \leq M^2$

The metric is asymptotically flat.

t: time; r:radial coord.; θ: latitudinal $[0,\pi]$; ϕ:longitudinal $[0,2\pi)$。

Kerr-(A)dS

Carter (1968):

$$ds^2 = \rho^2\Big(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta}\Big) + \frac{\sin^2\theta\,\Delta_\theta}{\rho^2}\Big(a dt - (r^2+a^2)\frac{d\phi}{\Xi}\Big)^2 - \frac{\Delta_r}{\rho^2}\Big(dt - a\sin^2\theta\frac{d\phi}{\Xi}\Big)^2,$$

$$\rho^2 = r^2 + a^2\cos^2\theta\,, \qquad \Xi = 1 + \tfrac{1}{3}\Lambda a^2$$

$$\Delta_r = (r^2+a^2)(1-\tfrac{1}{3}\Lambda r^2) - 2mr$$

$$\Delta_\theta = 1 + \tfrac{1}{3}\Lambda a^2\cos^2\theta$$

Generalize to higher dimensions

- Asymptotic Minkowski: Meyer and Perry (1986)
- Asymptotic (A)dS: $D = 5$ Hawking, Hunter and Robinson (1998)
- Asymptotic (A)dS: Arbitrary D, hep-th/0404008, hep-th/0409155.

How to construct: $D = 4$ example

Kerr-AdS$_4$:

$$ds^2 = \rho^2\Big(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta}\Big) + \frac{\sin^2\theta\,\Delta_\theta}{\rho^2}\Big(a dt - (r^2 + a^2)\frac{d\phi}{\Xi}\Big)^2 - \frac{\Delta_r}{\rho^2}\Big(dt - a\sin^2\theta\frac{d\phi}{\Xi}\Big)^2\,,$$

$$\rho^2 = r^2 + a^2\cos^2\theta\,, \qquad \Xi = 1+\tfrac{1}{3}\Lambda a^2$$

$$\Delta_r = (r^2 + a^2)(1-\tfrac{1}{3}\Lambda r^2) - 2mr$$

$$\Delta_\theta = 1 + \tfrac{1}{3}\Lambda a^2\cos^2\theta$$

First notice that Ξ is a constant and can be absorbed into ϕ; it is currently so chosen that ϕ has period 2π from global analysis.

How to construct: $D = 4$ example

If we set $m = 0$, the solution becomes AdS$_4$ in ellipsoidal coordinates.

$$\begin{aligned} d\bar{s}^2 &= \rho^2\Big(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta}\Big) + \frac{\sin^2\theta\,\Delta_\theta}{\rho^2}\Big(a dt - (r^2 + a^2)\frac{d\phi}{\Xi}\Big)^2 \\ &\quad - \frac{\Delta_r}{\rho^2}\Big(dt - a\sin^2\theta\frac{d\phi}{\Xi}\Big)^2, \end{aligned}$$

$$\rho^2 = r^2 + a^2\cos^2\theta\,, \qquad \Xi = 1 + \tfrac{1}{3}\Lambda a^2$$

$$\Delta_r = (r^2 + a^2)(1 - \tfrac{1}{3}\Lambda r^2)$$

$$\Delta_\theta = 1 + \tfrac{1}{3}\Lambda a^2\cos^2\theta$$

It might be somewhat strange that this complicated looking metric is simply (A)dS$_4$. (If $\Lambda = 0$, it is Minkowski spacetime.)

Coordinate transformation

Let $\Lambda = -3/\ell^2$.

$$\frac{(r^2+a^2)\sin^2\theta}{\Xi} = \rho^2 \sin^2\tilde\theta\,,$$
$$\frac{(1-\ell^{-2}a^2\cos^2\theta)(1+\ell^{-2}r^2)}{\Xi} = 1+\ell^{-2}\rho^2\,,$$

The metric becomes

$$ds^2 = (1+\ell^{-2}\rho^2)dt^2 + \frac{\rho^2}{1+\ell^{-2}\rho^2} + \rho^2\Big(d\tilde\theta^2 + \sin^2\tilde\theta\,(d\phi + \ell^{-2}a\,dt)^2\Big)\,.$$

AdS in rotating frame with angular velocity:

$$\Omega_\infty = -\ell^{-2}a\,.$$

Lesson to be learned

If we write the AdS_4 metric in the proper coordinates, turning the vacuum AdS_4 to rotating black hole is quite trivial. Of course, this is only a retrospective knowledge. However, we can now apply this method to the higher-dimensional construction.

Kerr-Shield form

The construction appears even more trivial in the Kerr-Shield form

$$ds^2 = d\bar{s}^2 + Uk^2\,, \qquad U = \frac{-2mr}{\rho^2}\,,$$

$$k = k_\mu dx^\mu = dt - a\sin^2\theta\frac{d\phi}{\Xi} - \frac{\rho^2 dr}{\Delta_r^0}\,,$$

Δ_r^0 is Δ_r with $m = 0$. The mass parameter m appears in the metric linearly!

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \eta_{\mu\nu}dx^\mu dx^\nu + \frac{2m}{U}(k_\mu dx^\mu)^2\,,$$

$$k = k_\mu dx^\mu = dt + \frac{r(xdx + ydy) + a(xdy - ydx)}{r^2 + a^2} + \frac{zdz}{r}\,,$$

and

$$U = r + \frac{a^2 z^2}{r^3}\,,$$

where r is defined by

$$\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1\,.$$

Comments on rotations

In flat space, the rotation generators are ${L^i}_j = x^i \partial_{x^j} - x^i \partial_{x^j}$. (They form the $SO(n)$ group.) They are thus rank-2 tensors, which happen to be dual to vectors in three dimensional space.

Thus in the three-dimensional space, there can be no two orthogonal planes. There can only be one independent rotation in $D = 3$ dimensions.

In the four-dimensional space (x_1, x_2, x_3, x_4), the plane (x_1, x_2) is orthogonal to that of (x_3, x_4). Thus there can be two independent orthogonal rotations in $D = 4 + 1$ spacetime.

In general, if the spacetime has dimensions D, there can be $N = [D - 1/2]$ orthogonal rotations, corresponding N number of rotation parameters a_i.

The Kerr metrics in general even D-dimensions (Myers-Perry):

$$k = k_\mu dx^\mu = dt + \sum_{i=1}^{n-1} \frac{r(x_i dx_i + y_i dy_i) + a_i(x_i dy_i - y_i dx_i)}{r^2 + a_i^2} + \frac{z dz}{r}\,,$$

with

$$U = \frac{1}{r}\Big(1 - \sum_{i=1}^{n-1} \frac{a_i^2\,(x_i^2 + y_i^2)}{(r^2 + a_i^2)^2}\Big) \prod_{j=1}^{n-1} (r^2 + a_j^2)\,,$$

and

$$\sum_{i=1}^{n-1} \frac{x_i^2 + y_i^2}{r^2 + a_i^2} + \frac{z^2}{r^2} = 1\,.$$

For odd D, drop z term and $U \to 1/rU$.

The Kerr-(A)dS metrics in D-dimensions: $ds^2 = d\bar{s}^2 + \frac{2M}{U}\,(k_\mu\, dx^\mu)^2\,,$ where the de Sitter metric $d\bar{s}^2$, the null 1-form k_μ, and the function $U(r,\mu_i)$ are given by

$$d\bar{s}^2 = -W\,(1-\lambda\, r^2)\, dt^2 + F\, dr^2 + \sum_{i=1}^{N+\epsilon} \frac{r^2+a_i^2}{1+\lambda\, a_i^2}\, d\mu_i^2$$
$$+ \sum_{i=1}^{N} \frac{r^2+a_i^2}{1+\lambda\, a_i^2}\, \mu_i^2\, d\phi_i^2 + \frac{\lambda}{W\,(1-\lambda\, r^2)} \Big(\sum_{i=1}^{N+\epsilon} \frac{(r^2+a_i^2)\, \mu_i\, d\mu_i}{1+\lambda\, a_i^2} \Big)^2 ,$$
$$k_\mu\, dx^\mu = F\, dr + W\, dt - \sum_{i=1}^{N} \frac{a_i\, \mu_i^2}{1+\lambda\, a_i^2}\, d\phi_i\,,$$
$$U = r^\epsilon \sum_{i=1}^{N+\epsilon} \frac{\mu_i^2}{r^2+a_i^2} \prod_{j=1}^{N} (r^2+a_j^2)\,,$$

where the functions $W(\mu_i)$ and $F(r,\mu_i)$ are defined to be

$$W \equiv \sum_{i=1}^{N+\epsilon} \frac{\mu_i^2}{1+\lambda\, a_i^2}\,, \qquad F \equiv \frac{1}{1-\lambda\, r^2} \sum_{i=1}^{N+\epsilon} \frac{r^2\, \mu_i^2}{r^2+a_i^2}\,.$$

hep-th/0404008, hep-th/0409155.

Black hole topology

Black hole topology describes the topology describe the black hole horizon. In four dimensions, it was shown to be 2-sphere. The spherically-symmetric and static Schwarzschild black hole must has round 2-sphere as its horizon. Rotating Kerr black holes is elliptic, but the topology remains 2-sphere.

New black topology in higher dimensions

In higher-dimensions, in addition to S^3 sphere topology, black hole can have other topologies as well, such as $S^2 \times S^1$. Such a black ring solution that is asymptotic to Mink_5 was constructed first by Emparan and Reall.
(Phys.Rev.Lett. 88 (2002) 101101)

How many do such black hole objects exist in higher dimensions? How to classify them? It remains illusive.

We shall come back to this point later.

Kerr-AdS-NUT: Plebanski (1975)

Let $u = a\cos\theta$, the Kerr-AdS can be written as follows

$$ds^2 = (x^2 + u^2)\Big(\frac{dr^2}{\Delta_r} + \frac{du^2}{\Delta_u}\Big) + \frac{\Delta_u}{r^2 + u^2}(dt - r^2 d\phi)^2 - \frac{\Delta_r}{r^2 + u^2}(dt + u^2 d\phi)^2\,,$$

$$\Delta_r = (a^2 + r^2)(1 - \tfrac{1}{3}\Lambda r^2) - 2mr$$

$$\Delta_u = (a^2 - u^2)(1 + \tfrac{1}{3}\Lambda u^2) - 2\ell r$$

- r and u are in the "equal" footing; the metric becomes more elegant
- Introducing a "NUT" parameter ℓ, which can be viewed as a "magnetic" dual of the mass
- Exist naked closed time-like circles (CTC)

General Kerr-AdS-NUT in D

$D = 2n$:

$$ds^2 = \sum_{\mu=1}^{n} \Big[\frac{U_\mu}{X_\mu} dx_\mu^2 + \frac{X_\mu}{U_\mu}(W_\mu dt - \sum_{i=1}^{n-1} \gamma_\mu^i d\phi)^2\Big]$$

$$U_\mu = \prod\nolimits_{\nu=1}^{\prime n}(x_\nu^2 - x_\mu^2)\,, \qquad X_\mu = -(1 - g^2 x_\mu^2)\prod_{k=1}^{n-1}(a_k^2 - x_\mu^2) - 2M_\mu x_\mu$$

$$W_\mu = \prod\nolimits_{\nu=1}^{\prime n}(1 - g^2 x_\mu^2)\,, \qquad \gamma_\mu^i = \prod\nolimits_{\nu=1}^{\prime n}(a_i^2 - x_\nu^2)$$

Analogous expression for $D = 2n + 1$. hep-th/0604125 $N = [\frac{1}{2}(D-1)]$ rotations, 1 mass and $(N-1)$ NUT parameters.

The key is to solve $\sum_i^{[D/2]} \mu_i^2 = 1$ by

$$\mu_i^2 = \frac{\prod_{\alpha=1}^{n}(a_i^2 - y_\alpha^2)}{\prod_{k=1}^{\prime n}(a_i^2 - a_k^2)}\,.$$

Plebanski-Demianski (1976)

In $D = 4$, the Kerr-AdS-NUT solution (Plebanski) can have an overall conformal factor:

$$d\tilde{s}^2 = \frac{1}{(1 - xy)^2} ds^2_{\text{Pleb}}$$

Here ds^2_{Pleb} means the Plebanski-type of ansatz, not the Plebanski metric. The functions X and Y in ds^2_{Pleb} can be determined as 4'th order polynomials of x and y respectively.

This is the most general metric of the Bianchi D metrics in four dimensions.

But we do not know yet how to generalize this to higher dimensions.

So far there is only some progress in generalization to $D = 5$ without a cosmological constant, which we shall review.

New $D = 5$ metric

$R_{\mu\nu} = -4\lambda g_{\mu\nu}$:
Starting from the rotating black hole:

$$ds_5^2 = \frac{x-y}{4X}dx^2 + \frac{y-x}{4Y}dy^2 + \frac{X(d\phi + yd\psi)^2}{x(x-y)} + \frac{Y(d\phi + xd\psi)^2}{y(y-x)}$$

$$+\frac{a_0}{xy}\Big(d\phi + (x+y)d\psi + xydt\Big)^2$$

$$X = a_0 + a_1 x + a_2 x^2 - \lambda x^3\,, \qquad Y = a_0 + b_1 y + a_2 y^2 - \lambda y^3$$

The metric has a scaling symmetry:

$$x \to \alpha x\,, \qquad y \to \alpha y$$

together with appropriate scaling of the parameters. This is analogous to the Plebanski metric, in which case, the scaling symmetry is broken by the Plebanski-Demianski generalization.

We find analogous generalization, provided that $\lambda = 0$.

New $D=5$ Ricci-flat metric 0804.1152

$$R_{\mu\nu} = 0:$$

$$ds_5^2 = \frac{1}{(1-xy)^2}\Big[\frac{x-y}{4X}dx^2 + \frac{y-x}{4Y}dy^2 + \frac{X(d\phi + y d\psi)^2}{x(x-y)} + \frac{Y(d\phi + x d\psi)^2}{y(y-x)}\Big]$$

$$+\frac{a_0}{xy}\Big(d\phi + (x+y)d\psi + xy dt\Big)^2$$

$$X = a_0 + a_3 x + a_2 x^2 + a_1 x^3 + a_0 x^4\,, \qquad Y = a_0 + a_1 y + a_2 y^2 + a_3 y^3 + a_0 y^4$$

Make a change of coordinates $x \to 1/x$, $t \to \mathrm{i}t$, $\phi \to \mathrm{i}\phi$ and $\psi \to \mathrm{i}\psi$, we have

$$ds^2 = \frac{1}{(x-y)^2}\Big[\frac{x(1-xy)dx^2}{4G(x)} - \frac{x(1-xy)dy^2}{4G(y)} - \frac{G(x)(d\phi + y d\psi)^2}{1-xy}$$

$$+\frac{xG(y)(d\psi + x d\phi)^2}{y(1-xy)}\Big] - \frac{a_0 y}{x}\Big(dt + \frac{x}{y}d\phi + (x + y^{-1})d\psi\Big)^2$$

$$G(\xi) = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3 + a_0\xi^4$$

Black-ring limit

Take a limit

$$x \to \epsilon^2 x, \quad y \to \epsilon^2 y, \quad \phi \to \epsilon\phi, \quad \psi \to \epsilon\psi, \quad t \to \epsilon^{-1} t$$

$$a_0 \to \epsilon^2 a_0 \quad a_1 \to a_1\,, \quad a_2 \to \epsilon^{-2} a_2, \quad a_3 \to \epsilon^{-4} a_3$$

with $\epsilon \to 0$, we have the black ring metric

$$\begin{aligned} ds^2 \;=\; & \frac{1}{(x-y)^2}\Big[\frac{x dx^2}{4G(x)} - \frac{x dy^2}{4G(y)} - G(x) d\phi^2 + \frac{x G(y) d\psi^2}{y}\Big] \\ & - \frac{a_0 y}{x}(dt + y^{-1} d\psi)^2 \end{aligned}$$

with

$$G(\xi) = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3$$

This metric of the black ring is somewhat simpler in the coordinates and parametrization used in solution obtained originally by Emparan and Reall, hep-th/0110260.

The global analysis of this black ring and the more general Ricci-flat metric can be found in arXiv:0804.1152

Lesson learned

The generalization of the Plebankski metric to higher dimensions has been done, but it does not give rise to anything particular interesting since Plebanski metrics has typically naked CTC except when reduced to the rotating black holes.

The generalization of the Plebanski-Demianski to higher dimensions can be very fruitful. The $D = 5$ example contains the Ricci-flat black ring solution.

If we only knows how to add the cosmological constant!

If we only knows how to generalize to $D > 5$!

Analytical solutions of black objects with $S^n \times S^m$ topology? It is so tantalizing!!!

Global Analysis: Black Ring

The black ring is significant since its horizon geometry is $S^2 \times S^1$, rather than the expected S^3.

You might think $S^2 \times S^1$ is easy since a direct product of $D = 4$ Schwarzschild black hole and a circle satisfies the Einstein equation in $D = 5$:

$$ds_5^2 = -f\,dt^2 + \frac{dr^2}{f} + r^2 d\Omega_2^2 + d\psi^2\,, \qquad f = 1 - \frac{2M}{r}\,.$$

Even though this metric's horizon geometry is $S^2 \times S^1$, we do not call this a black ring, because its asymptotic spacetime is $\text{Mink}_4 \times S^1$ rather than Mink_5. Again, black objects with different topologies should be discussed only in the context of the same asymptotic region.

In order for the above metric to be asymptotic Mink_5, ψ must be a real line, in which case the horizon geometry is $S^2 \times \mathbb{R}$, i.e. black (open) string!

black ring: global analysis

$$ds^2 = \frac{1}{(x-y)^2}\left[\frac{x dx^2}{4G(x)} - \frac{x dy^2}{4G(y)} - G(x) d\phi^2 + \frac{x G(y) d\psi^2}{y}\right]$$
$$- \frac{a_0 y}{x}(dt + y^{-1} d\psi)^2$$

with

$$G(\xi) = a_0 + a_1 \xi + a_2 \xi^2 + a_3 \xi^3$$

It can be expressed as

$$G(\xi) = -\mu^2 (\xi - \xi_1)(\xi - \xi_2)(\xi - \xi_3)\,, \qquad a_0 = \mu^2 \xi_1 \xi_2 \xi_3\,.$$

The metric has a scaling symmetry $(x, y) \to \alpha(x, y)$, implying that we can set $\mu = 1$. We therefore have three parameters left.

singularity and asymptotic flat region

The metric looks complicated, and not written in the form that we are familiar. Where is the asymptotic region and where is the singularity?

One way to find them out is by examining the curvature:

$$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{24(x-y)^4}{x^6}\Big(16\xi_1^2\xi_2^2\xi_3^2 - 4\left(\xi_1+\xi_2+\xi_3\right)x^4 y$$
$$+2\left(\xi_2\xi_3+\xi_1\left(\xi_2+\xi_3\right)\right)x^3(x-2y) + \Big(\left(3\xi_2^2+10\xi_3\xi_2+3\xi_3^2\right)\xi_1^2$$
$$+10\xi_2\xi_3\left(\xi_2+\xi_3\right)\xi_1 + 3\xi_2^2\xi_3^2\Big)x^2 - 4\xi_1\xi_2\xi_3 x^2(2x-3y)$$
$$+x^4\left(3x^2-4xy+8y^2\right) - 16\xi_1\xi_2\xi_3\left(\xi_2\xi_3+\xi_1\left(\xi_2+\xi_3\right)\right)x\Big).$$

Flat region: $x = y$

Singularities: $x = 0$, $x = \infty$, $y = \infty$.

$y = 0$ is not a singularity.

Signature

$$ds^2 = \frac{1}{(x-y)^2}\Big[\frac{x dx^2}{4G(x)} - \frac{x dy^2}{4G(y)} - G(x) d\phi^2 + \frac{x G(y) d\psi^2}{y}\Big] \\ -\frac{a_0 y}{x}(dt + y^{-1} d\psi)^2$$

with

$$G(\xi) = -\mu^2(\xi-\xi_1)(\xi-\xi_2)(\xi-\xi_3)\,, \qquad a_0 = \mu^2 \xi_1\xi_2\xi_3\,.$$

The scaling symmetry implies that we can set $\mu = 1$. Near the asymptotic region $x \sim y$:

$$G(x) < 0, \qquad x < 0, \qquad y < 0, \qquad G(y) > 0\,.$$

We thus select

$$\xi_1 < \xi_2 < 0 < \xi_3$$

with

$$x \in [\xi_1, \xi_2]\,, \qquad y \in [\xi_2, \infty)\,.$$

i.e. x is the compact coordinate, analogous to θ; y is non-compact, analogous to r.

Metric singularities

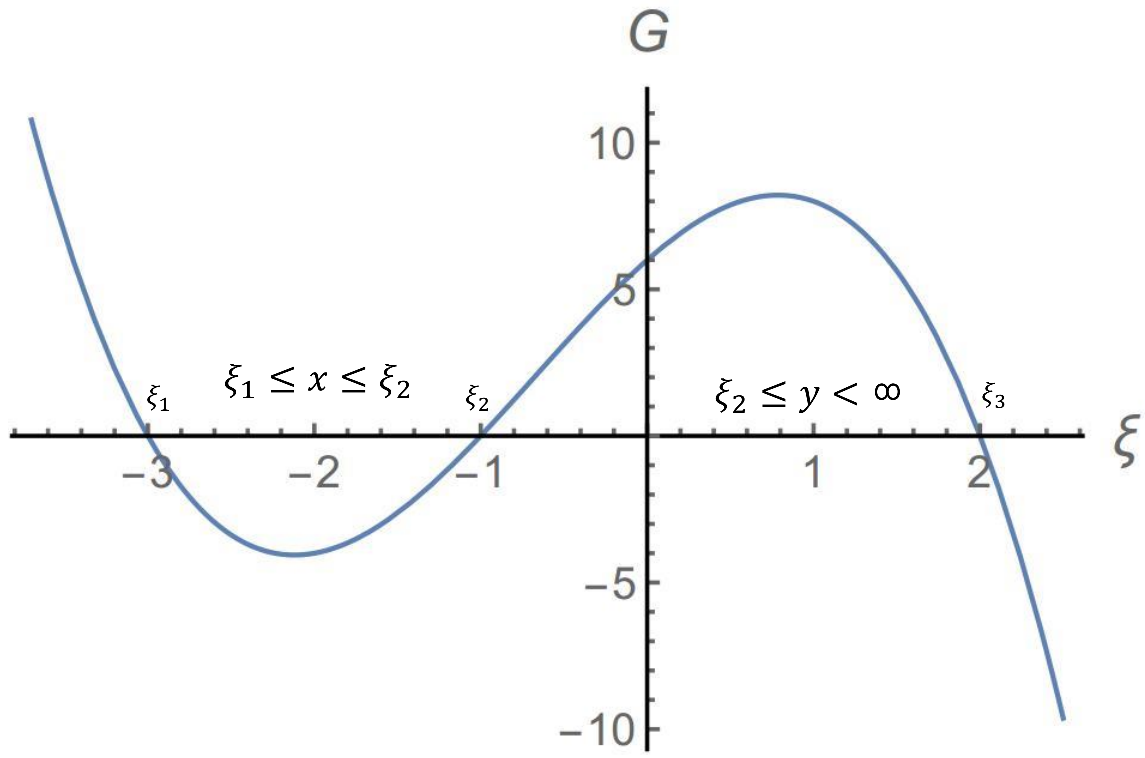
$$ds^2 = \frac{1}{(x-y)^2}\left[\frac{xdx^2}{4G(x)} - \frac{xdy^2}{4G(y)} - G(x)d\phi^2 + \frac{xG(y)d\psi^2}{y}\right]$$
$$-\frac{a_0 y}{x}(dt + y^{-1}d\psi)^2$$

We now look at the metric singularities (not curvature singularities) $x \to \xi_1, \xi_2, \xi_3$and $y \to \xi_1, \xi_2, \xi_3$

$y = 0$ is neither a metric nor a curvature singularity:

$$g_{\psi\psi} = -\frac{x^2 (y - \xi_2)(y - \xi_3) + \xi_1 \left(x^2 (\xi_3 - y) + \xi_2 \left(x^2 + \xi_3(y - 2x)\right)\right)}{x(x-y)^2}.$$

Spacetime regions



$x = \xi_2 = y$: asymptotically flat region

$y = 0$: ergo sphere

$y = \xi_3$: event horizon

$y = \infty$: singularity

Identifying the S^2

$$ds^2 = \frac{1}{(x-y)^2}\Big[\Big(\frac{x dx^2}{4G(x)} - G(x) d\phi^2\Big) - \frac{x dy^2}{4G(y)} + \frac{x G(y) d\psi^2}{y}\Big] - \frac{a_0 y}{x}(dt + y^{-1} d\psi)^2$$

Let $\xi_1 = -\eta_1^2$, $\xi_2 = -\eta_2^2$. Killing vector ∂/∂_ϕ is degenerate at $x = \xi_1$ and $x = \xi_2$, Normalising to unit Euclidean surface gravity, the degenerate Killing vectors are, respectively,

$$\ell_1 = \frac{\eta_1}{(\eta_1^2 - \eta_2^2)(\xi_3 + \eta_1^2)}\,\frac{\partial}{\partial\phi}\,, \qquad \ell_2 = \frac{\eta_2}{(\eta_1^2 - \eta_2^2)(\xi_3 + \eta_2^2)}\,\frac{\partial}{\partial\phi}\,.$$

Since these must each generate 2π rotations, it follows that the two prefactors must be equal, and hence we must require

$$\xi_3 = \eta_1 \eta_2\,.$$

Now we have only two free parameters left.

Identifying the S^1

The metric is degenerated also at $y = \xi_2$. The degenerated Killing vector

$$\ell = \partial_\psi + \frac{1}{\eta_2^2}\partial_t \,.$$

is periodic. In order to avoid having time involved in periodic coordinates, we make a coordinate shift:

$$t \to \frac{t}{(\eta_1\eta_2)^{3/2}} + \frac{\psi}{\eta_2^2}\,.$$

Now define $(\phi_1, \phi_2) = (\eta_1 - \eta_2)(\eta_1 + \eta_2)^2(\psi, \phi)$, we have

$$\ell_{x=\xi_1} = \frac{\partial}{\partial\phi_2} = \ell_{x=\xi_2}\,, \qquad \ell_{y=\xi_2} = \frac{\partial}{\partial\phi_1}\,,$$

all generate 2π period.

The two azimuthal angles are therefore (ϕ_1, ϕ_2).

Event horizon

The event horizon is located at $y = \xi_3$, where $G(\xi_3) = 0$. The null Killing vector is

$$\ell = \frac{\partial}{\partial_t} + \Omega_1 \frac{\partial}{\partial \phi_1}\,, \qquad \Omega_1 = \frac{\sqrt{\eta_2}\left(\eta_2^2 - \eta_1^2\right)}{\sqrt{\eta_1}}$$

$$T = \frac{1}{2\pi}\eta_2\left(\eta_1 + \eta_2\right)\,, \qquad S = \frac{\pi^2}{2\eta_2(\eta_1^2 - \eta_2^2)(\eta_1 + \eta_2)^3}\,.$$

Near horizon geometry

$$ds^2 = -\frac{\eta_1^2}{\left(\eta_1^2 - \eta_2^2\right)^2 x}(d\phi_1 + \omega_+ dt)^2 + \frac{1}{(x - \eta_1\eta_2)^2}\Bigg[\frac{d\phi_2^2\left(\eta_1^2 + x\right)(x - \eta_1\eta_2)\left(\eta_2^2 + x\right)}{(\eta_1 - \eta_2)^2(\eta_1 + \eta_2)^4} - \frac{x dx^2}{4\left(\eta_1^2 + x\right)(x - \eta_1\eta_2)\left(\eta_2^2 + x\right)}\Bigg] + \frac{x}{4\eta_1\eta_2(\eta_1 + \eta_2)^2(x - \eta_1\eta_2)^2}(d\rho^2 - \kappa^2\rho^2 dt^2)\,, \qquad T = \kappa/(2\pi)\,.$$

Asymptotic infinity

The asymptotic infinite is located at $x = \xi_2 = y$. To see it specifically, we set define

$$\frac{\sqrt{\xi_2 - x}}{y-x} = \frac{(\eta_1+\eta_2)\sqrt{\eta_1-\eta_2}}{\sqrt{\eta_2}}\, r\cos\theta\,,$$
$$\frac{\sqrt{y-\xi_2}}{y-x} = \frac{(\eta_1+\eta_2)\sqrt{\eta_1-\eta_2}}{\sqrt{\eta_2}}\, r\sin\theta\,.$$

The the $r \to \infty$ limit, we have

$$ds^2 \to -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi_1^2 + \cos^2\theta d\phi_2^2)\,.$$

It is indeed the Minkowski spacetime. The mass and angular momentum can be read off from the usual Komar integration

$$M = \frac{3\pi}{8\eta_2(\eta_1^2-\eta_2^2)(\eta_1+\eta_2)}\,, \qquad J_{\phi_2} = 0\,,$$
$$J_{\phi_1} = \frac{\pi\eta_1^{3/2}}{4\eta_2^{3/2}(\eta_1^2-\eta_2^2)^2(\eta_1+\eta_2)^2}\,.$$

$$dM = TdS + \Omega_{\phi_1} dJ_{\phi_1}\,, \qquad M = \tfrac{3}{2}(TS + \Omega_{\phi_1}J_{\phi_1})\,.$$

Further application:

Euclideanization and Einstein-Sasaki manifolds

Let us turn our attention to compact Euclidean spaces. Although Ricci-flat metrics can smoothly extend to a compact manifold, an analytical solution is more or less inconceivable. This is because it is hard to imagine to construct a metric with a Killing vector K_i, which satisfies

$$-\Box K_i - R_{ij} K^i = 0$$

Multiply by K^i and integrate over the manifold. For a compact manifold, integration by parts on the first term gives no boundary contribution, and hence one concludes

$$\int_{\mathcal{M}} (|\nabla_i K_j|^2 - R_{ij} K^i K^j) = 0$$

For the Ricci-flat compact metrics, it must be that $\nabla_i K_j$ pointwise everywhere in the manifold. Leaving aside the trivial possibility that there are flat S^1 factors, such a covariantlyconstant vector will not exist. Therefore there can be no Killing vectors in a non-flat Ricci-flat compact manifold.

However, if $R_{ij} = \lambda g_{ij}$ with positive λ, metrics of compact spaces can be analytically constructed, such as S^n.

Euclidean space with reduced holonomy: Gravitational Instanton

Let us consider cohomogeneity-one Ricc-flat space in four dimensions with the ansatz

$$ds_4^2 = d\rho^2 + a(\rho)^2(d\psi + \cos\theta d\phi)^2 + b(\rho)^2(d\phi^2 + \sin^2\theta d\phi^2)\,.$$

For constant ρ, it describe a homogeneous squashed S^3. This is the most general ansatz that preserves the $SU(2)\times U(1)$ isometry of the squashed S^3. $R_{\mu\nu} = 0$ imply

$$\begin{aligned} 0 &= \frac{a''}{a} + \frac{2a'b'}{ab} - \frac{a^2}{2b^4}\,, \\ 0 &= \frac{b''}{b} + \frac{b'^2}{b^2} + \frac{a'b'}{ab} + \frac{a^2}{2b^4} - \frac{1}{b^2}\,, \\ 0 &= -\frac{a''}{a} - \frac{2b''}{b}\,. \end{aligned}$$

A prime is a derivative with respect to ρ.

Solving this set of equations is not necessarily simple. Furthermore, we have only two functions (a, b), but three equations; are they consistent? Solving for (a'', b'') from the first and second equations, and substituting it the third, we have

$$H = \frac{4a'b'}{ab} + \frac{2b'^2}{b^2} + \frac{a^2}{2b^4} - \frac{2}{b^2} = 0\,.$$

This is in fact the Hamiltonian constraint. It is easy to verify that

$$H' = -\Big(\frac{2a'}{a} + \frac{4b'}{b}\Big)H\,.$$

Thus it is consistent to set the Hamiltonian to 0.

Lagrangian and Hamiltonian

Let us make a coordinate transformation $d\rho = ab^2 d\eta$, so that the metric is now

$$ds_4^2 = a^2 b^4 d\eta^2 + a^2(d\psi + \cos\theta d\phi)^2 + b^2(d\phi^2 + \sin^2\theta d\phi^2)\,.$$

The Hamiltonian is $H = T + V$ and the Lagrangian $L = T - V$, with

$$T = \frac{4\dot{a}\dot{b}}{ab} + \frac{2\dot{b}^2}{b^2}\,, \qquad V = \tfrac{1}{2}a^2(a^2 - 4b^2)\,.$$

Here a dot is a derivative with respect to η. Denote T as

$$T = \tfrac{1}{2} g_{ij}\dot{x}^i\dot{x}^j\,, \qquad x^i = \{a, b\}, \qquad i = 1, 2\,,$$

with

$$g_{ij} = \begin{pmatrix} 0 & \frac{4}{ab} \\ \frac{4}{ab} & \frac{4}{b^2} \end{pmatrix}, \qquad g^{ij} = \begin{pmatrix} -\frac{1}{4}a^2 & -\frac{1}{4}ab \\ -\frac{1}{4}ab & 0 \end{pmatrix}.$$

If there exists a function W, so that we can write the V as

$$V = -\tfrac{1}{2} g^{ij}\frac{\partial W}{\partial x^i}\frac{\partial W}{\partial x^j}\,.$$

Then the Lagrangian becomes

$$L = \tfrac{1}{2} g_{ij}\Big(\dot{x}^i - g^{ik}\partial_k W\Big)\Big(\dot{x}^j - g^{j\ell}\partial_\ell W\Big)\,.$$

This W is called superpotential. If it exists, then the equations of motion reduce to a set of first-order equations

$$\dot{x}^i = g^{ij}\partial_j W\,.$$

For our system, the W indeed exists, e.g.

$$W = a^2 + 2b^2\,.$$

This implies we have the first-order equations

$$a' = \frac{-a^2+2b^2}{2b^2}\,,\qquad b' = \frac{a}{2b}\,.$$

Here a prime is a derivative with respect to ρ again. It is easy to check that the above first-order equations solve the Einstein equation $R_{\mu\nu} = 0$. This equation can be easily solved, leading to

$$\begin{aligned} ds_4^2 &= \frac{dr^2}{U} + \frac{1}{4}r^2 U(d\psi + \cos\theta d\phi)^2 + \tfrac{1}{4}r^2(d\phi^2 + \sin^2\theta d\phi^2)\,. \\ U &= 1 - \frac{a}{r^4}\,. \end{aligned}$$

This is the Eguchi-Hanson instanton. (Question: there is another superpotential in this system, that would give Taub-NUT solution. Can you find it?)

Conifold and resolution

$T^{1,1}$ Einstein-space in five dimensions

$$ds_5^2 = \tfrac{1}{9}\sigma^2 + \tfrac{1}{6}d\Omega_2^2 + \tfrac{1}{6}d\tilde{\Omega}_2^2\,,$$

with $\sigma = d\psi + \cos\theta d\phi + \cos\tilde{\theta} d\tilde{\phi}$. It is Einstein with $R_{ij} = 4g_{ij}$. Conifold is

$$ds_6^2 = dr^2 + r^2 ds_5^2\,,$$

which is Ricci-flat, but with a singularity at $r = 0$. Resolution

$$ds_6^2 = \frac{r^2+6a^2}{r^2+9a^2}dr^2 + \frac{1}{9}\Big(\frac{r^2+9a^2}{r^2+6a^2}\Big)r^2\sigma^2 + \frac{1}{6}r^2 d\Omega_2^2 + \frac{1}{6}(r^2+6a^2)d\tilde{\Omega}^2\,.$$

Resolution: $r \to 0$, $\mathbb{R}^4 \times S^2$.

There is also resolution to give $\mathbb{R}^2 \times S^2 \times S^2$

and deformed conifold $\mathbb{R}^3 \times S^3$.

Einstein-Sasaki manifolds

$T^{1,1}$ is an earlier known example of Einstein-Sasaki manifolds, its cone gives the noncompact Ricci-flat Calabi-Yao metric.

Let the cosmological constant to be positive and the Kerr-AdS-NUT solution can then extend smoothly onto some compact manifold. In particular, if we further take some BPS limit such that the metric admits Killing spinors, we obtain in odd dimensions the Einstein-Sasaki spaces.

In $D = 5$, this leads to an infinite number of explicit and smooth Einstein-Sasaki metrics, including the $Y^{p,q}$ (hep-th/0403002) and L^{pqr} spaces (hep-th/0504225). They have very important application in the AdS/CFT correspondence, in the studying of the quiver gauge theory.

In even dimensions, the BPS limit will force the cosmological constant to vanish, leading to non-compact CY metrics with at most conical singularities (hep-th/0605222).

Summary so far

We have gone through large classes of explicit Einstein metrics $R_{\mu\nu} = \Lambda g_{\mu\nu}$ in diverse dimensions, focusing on rotating black holes.

After 100 years, it is unlikely to find any new Einstein metrics in four dimensions. (See "Exact Solutions to Einstein's Field Equations," second edition. Authors Stephani, Kramer, MacCallum, Hoenselaers and Herlt.)

However, there can still be exact solutions of intriguing black objects with non-trivial topologies in higher dimensions.

Charged black holes

Einstein-Maxwell theory:

$$\mathcal{L} = \sqrt{-g}(R - F_{\mu\nu}F^{\mu\nu})\,, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,.$$

Reissner-Nordström (RN)black holes

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega_2^2\,, \qquad f = 1 - \frac{2m}{r} + \frac{q^2}{r^2}\,,$$

with $A = \frac{q}{r}dt$.

With a cosmological constant

$$f \to -\tfrac{1}{3}\Lambda r^2 + 1 - \frac{2m}{r} + \frac{q^2}{r^2}\,.$$

The black hole has two parameters, mass m and charge q.

Kerr-Newman AdS solution

Charged AdS rotating black hole in four dimensions has long been known:

$$
\begin{aligned}
ds^2 &= \rho^2\Big(\frac{dr^2}{\Delta_r}+\frac{d\theta^2}{\Delta_\theta}\Big)+\frac{\Delta_\theta \sin^2\theta}{\rho^2}\Big(a dt-(r^2+a^2)\frac{d\phi}{\Xi}\Big)^2 \\
&\quad -\frac{\Delta_r}{\rho^2}\Big(dt-a\sin^2\theta\frac{d\phi}{\Xi}\Big)^2, \\
A &= \frac{q\,r}{\rho^2}\Big(dt-a\sin^2\theta\frac{d\phi}{\Xi}\Big)+\frac{p\cos\theta}{\rho^2}\Big(a dt-(r^2+a^2)\frac{d\phi}{\Xi}\Big),
\end{aligned}
$$

where,

$$
\rho^2=r^2+a^2\cos^2\theta\,,\qquad \Delta_\theta=1+\tfrac{1}{3}\Lambda a^2\cos^2\theta\,,\qquad \Xi=1+\tfrac{1}{3}\Lambda a^2\,,
$$
$$
\Delta_r=(r^2+a^2)(1-\tfrac{1}{3}\Lambda r^2)-2mr+p^2+q^2\,.
$$

Charged rotating black hole in higher dimensions

Although RN black hole, the charged static and spherically-symmetric solutions to Einstein-Maxwell theory, can be easily generalized to arbitrary higher dimensions, the exact solutions of rotating black hole in this theory may not exist beyond four dimensions.

Supergravity can have: a 5-D example

EM-gravity theory in $D = 5$:

$$\mathcal{L} = \sqrt{-g}(R - \tfrac{1}{4}F^2)\,.$$

EM-supergravity in $D = 5$

$$\mathcal{L} = \sqrt{-g}(R - \tfrac{1}{4}F^2) + \frac{1}{12\sqrt{3}}\epsilon^{\mu\nu\rho\sigma\lambda}F_{\mu\nu}F_{\rho\sigma}A_\lambda\,.$$

EM-gauged supergravity in $D = 5$:

$$\mathcal{L} = \sqrt{-g}(R - 2\Lambda - \tfrac{1}{4}F^2) + \frac{1}{12\sqrt{3}}\epsilon^{\mu\nu\rho\sigma\lambda}F_{\mu\nu}F_{\rho\sigma}A_\lambda\,.$$

Charged rotating black holes were constructed in $D = 5$ gauged supergravities [hep-th/0506029, 1108.4159]

Solution-generating tech: Applying Global Symmetry of String Theory

A fact is that there is no known example of analytical charged rotating black hole in Einstein-Maxwell theory beyond four dimensions.

A fact is that in supergravity, charged rotating black holes can be constructed!

What's the difference? Supergravity theories have additional global symmetry in Kaluza-Klein reduction. One can generate a new solution by applying global symmetry. This leads to a very useful solution-generating technique.

A topic that I shall discuss in August School in North-East University in Nanjing.

Constraints from the energy conditions

We have so far looked at a large number of known solutions to Einstein's field equations, based on some specific fundamental matter theories. We somehow get some general impressions such as

- Black holes have singularities
- Gravity is attractive

But how general are such statements? Gravity can be repulsive if the mass in the Schwarzschild black hole is negative.

In Einstein's theory of gravity, any spacetime metric $g_{\mu\nu}$ is possible from Einstein's equation $G_{\mu\nu} = T_{\mu\nu}$.

However, not all the $T_{\mu\nu}$ are allowed based on our understanding of matter theories.

Therefore the conditions on $T_{\mu\nu}$ will give restrictions on the possible spacetime geometries.

Preliminary: energy conditions

In Einstein gravity, you can get *any* spacetime since

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu}\,,$$

unless you have some redlines that you wont cross, namely the energy conditions on the $T^{\mu\nu}$.

For this talk, we shall focus on the spherically-symmetric and static metrics, with

$$T^{\mu}{}_{\nu} = \mathrm{diag}\{-\rho, p_1, p_2, p_3\} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p_1 & 0 & 0 \\ 0 & 0 & p_2 & 0 \\ 0 & 0 & 0 & p_3 \end{pmatrix}.$$

There can be various energy conditions:

- NEC: $\rho + p_i \geq 0$, for all $i = 1, 2, 3$;
- WEC: NEC & $\rho \geq 0$;
- DEC: NEC & $\rho - p_i \geq 0$, for all $i = 1, 2, 3$;
- SEC: NEC & $\rho + p_1 + p_2 + p_3 \geq 0$;
- TEC: $\rho - p_1 - p_2 - p_3 \geq 0$.

We thus have

$$\text{DEC} \supset \text{WEC} \supset \text{NEC}\,, \qquad \text{SEC} \supset \text{NEC}\,,$$

but DEC, SEC and TEC are pair-wise independent.

NEC is our ultimate redline.

Both the Schwarzschild and RN black holes satisfy *all* these energy conditions.

Examples: bouncing universe and wormholes

Bouncing cosmology $ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2)$

$$\rho = -G^t_t\,, \qquad p = G^x_x\,, \qquad \rho + p = \frac{2\dot{a}}{a^2} - \frac{2\ddot{a}}{a}\,.$$

Bouncing ($\dot{a} = 0, \ddot{a} > 0$); therefore it violates the NEC; but it is ok for crunching ($\dot{a} = 0, \ddot{a} < 0$).

Wormhole: $ds^2 = -dt^2 + dr^2 + (r^2 + a^2)d\Omega_2^2$

$$\rho = -G^t_t\,, \qquad p_r = G^r_r\,, \qquad \rho + p_r = -\frac{2a^2}{\left(a^2 + r^2\right)^2}\,.$$

It violates the NEC.

Penrose entropy bound

In the 70's, Penrose proposed a rather simple looking inequality

$$2M_{\mathrm{ADM}} \geq \sqrt{\frac{A[\sigma]}{4\pi}}\,,$$

where $A[\sigma]$ is the minimal area enclosing the apparent horizon σ.

In my view, the statement is only becoming significant for black holes.

Proof becomes simpler for assuming the spherical symmetry:

- Static black holes: the sufficient condition is WEC. [Bray (2001); Huisken, Ilmanen (2001)]
- Dynamic black holes: DEC. [Hayward,gr-qc/9408002]

The belief is that Schwarzschild black hole is the only one that saturates the bound.

Why is it important? For stationary black holes, $A = 4S$.

The Penrose inequality then becomes an entropy bound for a system of given total energy.

Spherically-symmetric and static

Spherically-symmetric and static metric:

$$ds^2 = -f(r)e^{-\chi(r)}dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 .$$

The Einstein field equation $G_{\mu\nu} = 8\pi T_{\mu\nu}$ is solved by

$$f' = \frac{1 - 8\pi r^2 \rho - f(r)}{r} , \qquad \chi' = -\frac{8\pi r(\rho + p_r)}{f} ,$$

$$p_r' = \frac{\rho - 3p_r + 4p_T}{2r} - \frac{(\rho + p_r)(1 + 8\pi p_r r^2)}{2rf} ,$$

where $p_r = p_1$ and $p_T = p_2 = p_3$. Asymptotic boundary conditions:

$$f(r) = 1 - \frac{2M}{r} + \cdots , \qquad \lim_{r\to\infty} r\,\chi(r) = 0 .$$

Immediately, the NEC implies that $\chi \geq 0$ and $e^{-\chi} < 1$.

If r_+ is the horizon $f(r_+) = 0$, we have

$$T = \left. \frac{e^{-\frac{1}{2}\chi} f'}{4\pi} \right|_{r=r_+} , \qquad S = \pi r_+^2 .$$

Quasi-local mass

Hawking-Geroch mass

$$m(r) = \frac{r}{2}(1 - f)\,, \qquad m(r_+) = \tfrac{1}{2} r_+\,, \qquad m(\infty) = M\,.$$

Using the three equations of motion, it is easy to establish

$$m' = 4\pi r^2 \rho\,.$$

Thus if imposing WEC, $\rho \geq 0$, then $m(r)$ is a monotonically nondecreasing function, and hence

$$m(\infty) \geq m(r_+)\,, \qquad 2M \geq r_+ = \sqrt{\frac{S}{\pi}}\,.$$

So for the spherically-symmetric and static black holes, the Penrose inequality is extremely easy to prove and the sufficient condition is WEC.

Equality vs Inequality

A black hole is typically specified by two sets of quantities

- asymptotic quantities: M, J, Q, *etc.*
- horizon data: T, S, Ω_+, Φ_+, *etc.*

This can leads to equalities between

- Differential: the first law $dM = TdS + \Omega_+ dJ + \Phi_+ dQ + \cdots$
- Algebraic: the Smarr relation $M = 2TS + 2\Omega_+ J + \Phi_+ Q + \cdots$

The Penrose inequality thus estimates two purely geometric quantities: one at asymptotic infinity and one on the horizon

$$2M \geq \sqrt{\frac{S}{\pi}}\,.$$

But geometric quantities involves (M, Ω, J, T, S). Therefore, there should be some inequalities associated with these geometric quantities under the some general energy conditions.

Bounds on the temperature

Based on various energy conditions, we find for static and spherically symmetric black holes that are asymptotic flat, there is a bound on temperature

$$\sqrt{\frac{1}{\pi\, S}} - \frac{M}{S} \le 2T \le \frac{3M}{S} - \sqrt{\frac{1}{\pi S}}\,.$$

Hossein Khodabakhshi will give a detail proof of this on Wednesday afternoon.

Exact scalar hairy black holes: engineering a scalar potential

Minimally-coupled scalar

$$\mathcal{L} = \sqrt{-g}(R - \tfrac{1}{2}(\partial\phi)^2 - V(\phi))\,.$$

Equations of motion

$$\begin{aligned} \delta\phi: &\quad \Box\phi = \frac{\partial V}{\partial\phi}\,, \\ \delta g^{\mu\nu}: &\quad G_{\mu\nu} = \tfrac{1}{2}(\partial_\mu\phi\partial_\nu\phi - \tfrac{1}{2}(\partial\phi)^2 g_{\mu\nu}) - \tfrac{1}{2}V g_{\mu\nu}\,. \end{aligned}$$

"No-go theorem?"

There is a rumour of no-go theorem about scalar black holes. This is in fact the easiest no-go theorem to get around in anyone's life. Let us consider the minimally-coupled scalar, whose equation of motion is

$$\Box\phi = \frac{dV}{d\phi}\,,$$

where V is the scalar potential. Assuming a static black hole, with the most general metric

$$ds^2 = -\lambda^2 dt^2 + g_{ij} dx^i dx^j\,.$$

where λ and g_{ij} are functions of x. Multiplying the scalar equation with ϕ and integrating over the spatial section, we have

$$\int D_i(\lambda\phi D^i\phi) - \lambda D_i\phi D^i\phi - \phi\frac{dV}{d\phi} = 0\,,$$

Thus we must have

$$\phi\frac{dV}{d\phi} \le 0\,,$$

which cannot be satisfied by a free scalar. But any scalar with concave potential can evade this no-go theorem.

Ansatz and eoms

Spherically-symmetric and static

$$ds^2 = -h(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{n-2}^2\,, \qquad \phi = \phi(r)\,,$$

equations of motion [1408.1514]

$$\frac{h''}{h} - \frac{h'^2}{2h^2} + \frac{f'h'}{2fh} + \frac{(n-3)h'}{rh} - \frac{f'}{rf} - \frac{2(n-3)(f-1)}{r^2 f} = 0\,,$$

$$\phi'^2 = \frac{(n-2)(fh' - hf')}{rfh}\,,$$

$$\frac{fh''}{h} - \frac{fh'^2}{2h^2} + \frac{f'h'}{2h} + \frac{(n-1)fh'}{rh} + \frac{f'}{r} + \frac{2(n-3)(f-1)}{r^2} + \frac{4V}{n-2} = 0\,.$$

Note that $V = V(\varphi(r))$. All the above equations are Einstein, and the scalar equation is automatically satisfied.

For a random given scalar potential $V(\phi)$, it is unlikely to have exact solutions. However, if we do not care the detail of $V(\phi)$, we can treat $V(\phi)$ as an unknown variable to be solved.

The first two equations does not involve V, and hence we can solve (h, f) after making a suitable ansatz for ϕ such as $\phi = q_1/r$. If this can be done indeed, we can then solve for $V = V(r)$ from the third equation, and convert back to $V = V(\phi)$.

This procedure was spelled out in detail in [1204.2720, 1308.1693, 1312.5374]. It is important to note that the parameter mass in the metric should not appear explicitly in $V = V(\phi)$; otherwise, the construction is not valid.

Reverse engineering: NLED

Nonlinear electrodynamics

$$\mathcal{L} = \sqrt{-g} f(F^2)\,, \qquad F^2 = F^{\mu\nu} F_{\mu\nu}\,, \qquad F = dA\,.$$

Maxwell theory $f = -F^2$. The variation of the Maxwell potential A_μ gives

$$\nabla_\mu(\varphi F^{\mu\nu}) = 0\,, \qquad \varphi(F^2) = f'(F^2)\,.$$

The energy-momentum tensor comes from the variation of the metric $g_{\mu\nu}$ is

$$T_{\mu\nu} = \tfrac{1}{2}\Big(-4\varphi F^2_{\mu\nu} + g_{\mu\nu} f(F^2)\Big)\,.$$

Einstein-NLED:

$$\mathcal{L} = \sqrt{-g}\Big(R + f(F^2)\Big)\,.$$

Spherically-symmetric and static metrics

In four dimensions, spherically-symmetric and static metrics can be sourced by both electric and magnetic charges [Bronnikov,gr-qc/0006014]

$$\begin{aligned} ds^2 &= -h(r)e^{-\sigma(r)}dt^2 + \frac{dr^2}{h(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \\ F &= \psi(r)dt \wedge dr + p\sin\theta d\theta \wedge d\phi\,. \end{aligned}$$

Einstein equations:

$$-G_t{}^t + G_r{}^r = -\frac{h}{r}\sigma' = 0\,, \quad \to \quad \sigma = 0\,.$$

Thus

$$F^2 = -2\psi^2 + \frac{2p^2}{r^4}\,, \quad \psi = \frac{q}{\varphi r^2}\,, \quad \varphi(F^2) = f'(F^2)\,.$$

$$G_t{}^t + G_r{}^r = \frac{2(rh' + h - 1)}{r^2} = f(F^2) + 4\varphi\psi^2.$$

Note that the right-hand side is independent of h.

Magnetic monopoles and reverse engineering technique

For magnetic solutions, the complete set of equations reduces to

$$\frac{2(rh' + h - 1)}{r^2} = f(F^2)\,, \qquad F^2 = \frac{2p^2}{r^4}\,.$$

Reverse engineering technique: [Fan, Wang 1610.02636]

$$f(F^2) = \frac{2(rh' + h - 1)}{r^2}\Big|_{r\to(\frac{2p^2}{F^2})^{\frac{1}{4}}}.$$

This shows that any special static metric with $g_{tt}g_{rr} = -1$ can be viewed as a magnetic monopole of certain NLED.

Regular black holes

Penrose's singularity theorem actually does not say much about the inside of the event horizon of either a static or stationary black hole. [Hawking, Ellis; Y. Choquet-Bruhat; Senovilla, Garfinkle]

There is no proof under SEC of non-existence of such regular black holes, but there is no known counter example either.

Explicit examples have been constructed by relaxing the SEC, e.g. Bardeen, Hayward, and they satisfy WEC.

The first known example of regular black holes was perhaps sourced by the quasi-topological electromagnetism under special limits, but I am not quite so sure. [Liu, Mai, Li, Lü, 1907.10876; Cisterna,Giribet,Oliva, Pallikaris, 2004.05474]

Recently there is a review paper on regular black holes, which contain an extensive list of references. [Lan, Yang, Guo, Miao, 2303.11696]

Spherically-symmetric and static metrics

I shall discuss only the spherically-symmetric and static solutions.

The most general class:

$$ds^2 = -h(r)e^{-\sigma(r)}dt^2 + \frac{dr^2}{h(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,,$$

The special class $g_{tt}g_{rr} = -1$:

$$ds^2 = -h(r)dt^2 + \frac{dr^2}{h(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,,$$

The curvature singularity typically resides at $r = 0$. To avoid, one may try

- wormhole [Ellis]
- black bounce [Simpson, Visser, 1812.07114]
- dark wormhole that connects to dS spacetime [Geng, Lü, 1511.03681]

All violates the NEC in the frame work of Einstein gravity

A regular star-like core

Our Sun is regular and its core has constant g_{tt} and g_{rr} at the center (ignoring rotation):

$$h = 1 + a_2 r^2 + a_3 r^3 + \cdots, \; he^{-\sigma} = b_0 + b_2 r^2 + b_3 r^3 + \cdots, \quad b_0 > 0\,.$$

The SEC and DEC conditions imply that

$$\begin{aligned} \text{SEC}: &\quad b_2 > \max\{0, a_2 b_0\}\,; \\ \text{DEC}: &\quad a_2 < 0\,, \qquad a_2 b_0 < b_2 < -2 a_2 b_0\,. \end{aligned}$$

However for special static metrics:

$a_2 < 0 \to$ dS core; $\quad a_2 = 0 \to$ Mink core; $\quad a_2 > 0 \to$ AdS core.

The WEC or DEC requires a dS core with $a_2 < 0$, but the SEC requires an AdS core with $a_2 > 0$.

If the core is Minkowski with $a_2 = 0$, the same argument proceeds with a_3, and so on.

A regular core of a special static metric does not necessarily violate either DEC or SEC, but it does violate WEC $\oplus$ SEC.

Something weird about regular special static core

With $g_{tt}g_{rr} = -1$, the regular core implies that the time ticking rate at the core is the same as that at the asymptotic infinity.

This is counterintuitive since we would expect that there is gravitational time dilation at the center of our Sun.

We shall prove later that such a regular metric with flat asymptotic infinity will necessarily violate SEC.

Well-known examples

Both Bardeen and Hayward metrics are special static with

$$\text{Bardeen}: \qquad h = 1 - \frac{2Mr^2}{(r^2+g^2)^{\frac{3}{2}}}\,,$$

$$\text{Hayward}: \qquad h = 1 - \frac{2Mr^2}{r^3+g^3}\,,$$

Both violate DEC and SEC, but satisfy WEC. Both have dS core.

Regular metrics are fine-tuned objects

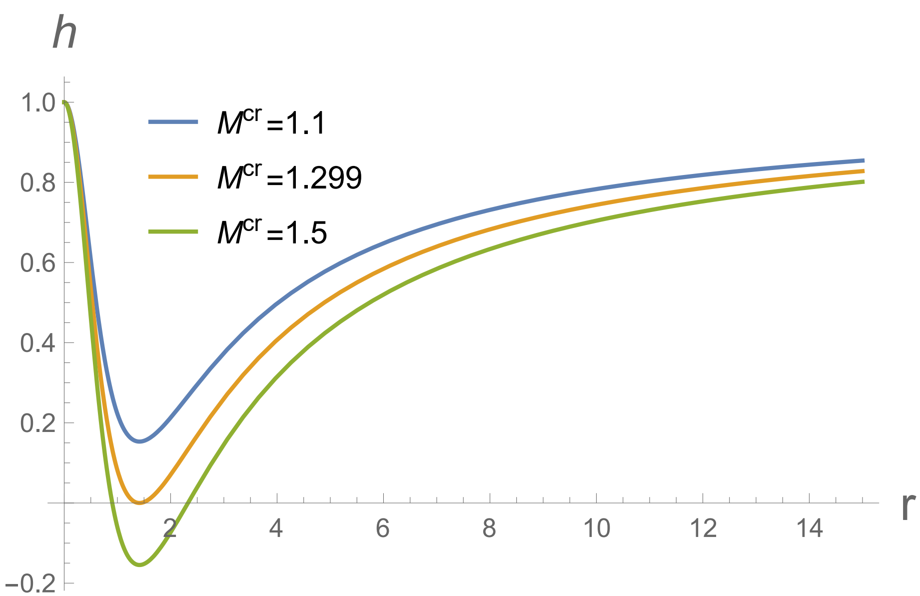
Let us consider an example, the Bardeen metric, which is a solution to Einstein's equation

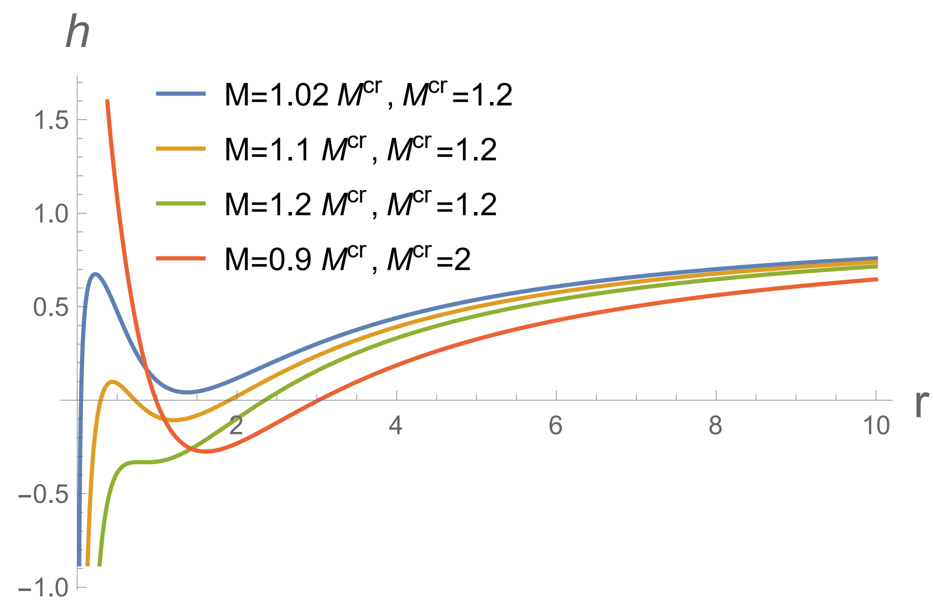
$$-G^t_t = -\frac{rh' + h - 1}{r} = \rho = \frac{6g^2 M^{\text{cr}} r}{\left(g^2 + r^2\right)^{5/2}}.$$

The most general solution is

$$h = 1 - \frac{2(M - M^{\text{cr}})}{r} - \frac{2M^{\text{cr}} r^2}{\left(g^2 + r^2\right)^{3/2}}.$$

The regular solution arises as $M = M^{\text{cr}}$.





Regular black holes as magnetic monopoles

Both Bardeen and Hayward metrics have $g_{tt}g_{rr} = -1$. Any such a metric can be viewed as a magnetic monopole of certain $f(F^2)$. Therefore

$$\text{Bardeen}: \qquad h = 1 - \frac{2Mr^2}{(r^2+g^2)^{\frac{3}{2}}}, \rightarrow f = \frac{1}{\alpha}\frac{(\alpha F^2)^{\frac{5}{4}}}{\left(1+\sqrt{\alpha F^2}\right)^{\frac{5}{2}}},$$

$$\text{Hayward}: \qquad h = 1 - \frac{2Mr^2}{r^3+g^3}, \rightarrow f = \frac{1}{\alpha}\frac{(\alpha F^2)^{\frac{3}{2}}}{\left(1+(\alpha F^2)^{\frac{3}{4}}\right)^2}.$$

[Ayon-Beato, Garcia, gr-qc/0009077; Fan, Wang 1610.02636

But the result is disappointing because of the fractional powers acting directly on F^2.

Regular magnetic monopoles from analytic $f(F^2)$

For magnetic monopoles, $F^2 = 2p^2/r^4$, we therefore require $f(0) = 0$ for asymptotic flatness, and $f(\infty)$ =finite for a regular core. We also require that in the weak-field limit:

$$f = -F^2 + \alpha_1 (F^2)^2 + \alpha_2 (F^2)^3 + \cdots .$$

Examples satisfy DEC include:

$$f = \frac{1}{\nu\alpha}\Big(\frac{1}{(1+\alpha F^2)^\nu} - 1\Big), \qquad f = -\frac{F^2}{(1+\alpha (F^2)^\mu)^\nu},$$

$$h = 1 - \frac{2M}{r} + \frac{r^2}{6\nu\alpha}\Big({}_2F_1[-\tfrac{3}{4},\nu;\tfrac{1}{4};-\alpha\tfrac{2p^2}{r^4}] - 1\Big),$$

$$h = 1 - \frac{2M}{r} + \frac{p^2}{r^2}{}_2F_1[\tfrac{1}{4\mu},\nu;1+\tfrac{1}{4\mu};-\alpha(\tfrac{2p^2}{r^4})^\mu].$$

$$f = \frac{1}{2\alpha}(e^{-2\alpha F^2} - 1), \qquad h = 1 - \frac{2\mu}{r} + \frac{r^2}{48\alpha}\Big(E_{\frac{7}{4}}(\frac{4\alpha p^2}{r^4}) - 4\Big).$$

Minkoski core:

$$f = \frac{1}{2\alpha}(e^{-2\alpha F^2} - 1)e^{-2\alpha F^2}.$$

Geodesic Completeness

Compare

$$\text{Bardeen}: \qquad h = 1 - \frac{2Mr^2}{(r^2+g^2)^{\frac{3}{2}}},$$
$$\text{Hayward}: \qquad h = 1 - \frac{2Mr^2}{r^3+g^3},$$

Both have dS core at $r = 0$. But recent work shows that Hayward is not regular; it has singularity at $r = -g$, and the dS core is not geodesically complete. [Zhou, Modesto, 2208.02557]

Regular metrics constructed from our analytic $f(F^2)$ are guaranteed that $h(r)$ is an even function of r:

$$h(-r) = h(r)\,.$$

The minus r region is identical to the positive r region, and hence can be identified so that the geodesic is complete in $r \in (0, \infty)$.

No-Go theorems

All our examples satisfy the DEC, but violate SEC. We now show that

- regular special static metrics must violate SEC;
- such metrics with Minkowski core must violate NEC.

We have seen that special static metrics must be able to do reverse engineering, with

$$\rho = -p_r = -\tfrac{1}{2} f(\chi)\,, \quad p_T = \tfrac{1}{2} f(\chi) - \chi f'(\chi)\,, \quad \chi = \frac{2p^2}{r^4} > 0\,.$$

The SEC requires

$$\rho + p_T = -\chi f'(\chi) \geq 0\,, \quad p_T = \tfrac{1}{2} f(\chi) - \chi f'(\chi) \geq 0\,.$$

The asymptotic flatness at large r requires $f(0) = 0$ and regularity at $r = 0$ requires that $f(\infty)$ be a constant. In this case $\chi f'(\chi)$ must vanish as $\chi \to \infty$.

Since $f'(\chi)$ must be negative due to the NEC, $f(\infty)$ must be a negative constant, indicating that the core is dS-like. Thus, the strong energy condition must be violated at the core.

In order to have a Minkowski core, we must have $f(\infty) = 0$. Thus for non-vanishing f, there must be a minimum of $f(\chi)$ and hence we must also have $f'(\chi) > 0$ at a certain spacetime region. This immediately leads to a violation of the NEC in the transverse direction.

The no-go theorems do not apply to the general static metrics.

2303.16924

BUT

- all the examples are magnetically charged.
- $F^2 = 2p^2/r^4$ is not regular at the core. For a purist, this is not acceptable, even though the metric is regular.
- Direct construction of electrically-charged regular black holes is not easy, and no-go theorems were claimed. [Bronnikov,gr-qc/0006014; Bokulić, Smolić, Jurić, 2206.07064]

But the Maxwell equation is

$$\nabla_\mu(\varphi F^{\mu\nu}) = 0\,, \qquad \varphi = f'(F^2)\,.$$

It suggests a dual field strength $G = dB$ such that

$$\varphi F = *G\,, \qquad \to \qquad \varphi^2 F^2 = -G^2\,.$$

But even if we can solve F^2 in terms of G^2, we cannot simply substitute it into the Lagrangian $f(F^2)$ to get its dual theory.

Scalar-vector and EM duality

$$\mathcal{L} = \sqrt{-g} f(F^2) \quad \Leftrightarrow \quad \mathcal{L} = \sqrt{-g}(-\phi^2 F^2 - V(\phi)) \,,$$

where

$$\phi^2 = -f'(\chi) \,, \qquad V = \chi f'(\chi) - f(\chi) \,.$$

Examples:

$$f = -\frac{F^2}{(1+(\alpha F^2)^n)^{\frac{1}{n}}} \,, \ \leftrightarrow \ V = \frac{1}{\alpha}\left(1 - \phi^{\frac{2n}{n+1}}\right)^{\frac{n+1}{n}} \,,$$

$$f = \frac{1}{\nu\alpha}\left(\frac{1}{(1+\alpha F^2)^\nu} - 1\right) , \ \leftrightarrow \ V = \frac{1}{\alpha\nu}\left(1 + \nu\phi^2 - (\nu+1)\phi^{\frac{2\nu}{\nu+1}}\right) ,$$

$$f = \frac{1}{2\alpha}(e^{-2\alpha F^2} - 1) \,, \ \leftrightarrow \ V = \frac{1}{2\alpha}(1 - \phi^2 + \phi^2 \log \phi^2) \,.$$

EM duality:

$$\mathcal{L} = \sqrt{-g}\Big(- \phi^2 F^2 - V(\phi)\Big) \quad \leftrightarrow \quad \mathcal{L} = \sqrt{-g}\Big(- \phi^{-2} G^2 - V(\phi)\Big).$$

Regular electrically-charged black holes

Consider the simplest example of a magnetic monopole:

$$
\begin{aligned}
f &= -\frac{F^2}{1+\alpha F^2}\,, \qquad \leftrightarrow \qquad V = \frac{1}{\alpha}(\phi-1)^2\,,\\
h &= 1 - \frac{2M}{r} + \frac{p^2}{r^2}\,{}_2F_1[\tfrac{1}{4},1;\tfrac{5}{4};-\frac{2\alpha p^2}{r^4}]\,.
\end{aligned}
$$

Metric can be regular, but $F^2 = 2p^2/r^4$ diverges at the $r=0$ core.

The same metric can be sourced by the electric field G of its dual theory, with

$$
G^2 = -\frac{F^2}{(1+\alpha F^2)^4} = \frac{2p^2 r^{12}}{\left(2\alpha p^2 + r^4\right)^4}\,,
$$

which is clearly regular from $r=0$ all the way to $r=\infty$.

"More" regular than the magnetic monopole.

EM duality symmetry

Maxwell theory is self-dual in four dimensions, but the general $f(F^2)$ is clearly not. Perturbatively in the weak-field limit, we find

$$\mathcal{L} = \sqrt{-g}\Big(- F^2 + \alpha (F^2)^2 + \beta (F^2)^3 + \cdots \Big),$$

is dual to

$$\mathcal{L} = \sqrt{-g}\Big(- G^2 + \alpha (G^2)^2 - (\beta + 4\alpha^2)(G^2)^3 + \cdots \Big).$$

In general, the $f(F^2)$ is self-dual, provided that the scalar potential its scalar-vector theory

$$\mathcal{L} = \sqrt{-g}\Big(- \phi^2 F^2 - V(\phi)\Big).$$

has the property that $V(\phi) = V(\phi^{-1})$.

EM self-dual NLED theories

Two well-known examples: Maxwell and BI theories.

Example 1: [Bronnikov 1708.08125]

$$V = \frac{1}{\alpha}(\phi^2 + \phi^{-2} - 2)\,, \quad \rightarrow \quad f = \frac{2}{\alpha}\Big(1 - \sqrt{1 + \alpha F^2}\Big)\,.$$

$$h = 1 - \frac{2M}{r} + \frac{r^2}{3\alpha}\left(1 - F_1\left(-\frac{3}{4}; -\frac{1}{2}, -\frac{1}{2}; \frac{1}{4}; -\frac{2p^2\alpha}{r^4}, -\frac{2q^2\alpha}{r^4}\right)\right)\,.$$

Example 2:

$$V = \frac{1}{2\alpha} - \alpha(\phi^2 + \phi^{-2})^{-1}\,, \qquad \rightarrow$$

$$f = \frac{1}{8\alpha}\left(\sqrt{1 + \sqrt{1 + 8\alpha F^2}}\left(3 - \sqrt{1 + 8\alpha F^2}\right)^{3/2} - 4\right),$$

$$h = 1 - \frac{2M}{r} + \frac{p^2 + q^2}{r^2} - \frac{2\alpha\left(p^2 - q^2\right)^2}{5r^6} + \frac{8\alpha^2\left(p^2 - q^2\right)^2\left(p^2 + q^2\right)}{9r^{10}} + \mathcal{O}(r^{-11}).$$

Example 3:

$$f = \frac{24}{\alpha} - \frac{8\sqrt{2}}{\alpha}(u + \frac{1}{u})\,, \qquad u = \sqrt{1 + \sqrt{1 + \alpha F^2}}.$$

Conclusions on regular black holes from NLED

- Illustrated that regular black holes are all fine-tuned objects.
- Established two no-go theorems on regular special static metrics ($g_{tt}g_{rr} = -1$)
  - it must violate the SEC
  - such a metric with Minkowski core must violate the NEC
- Obtained regular metrics as magnetic monopoles from analytic $f(F^2)$, but $F^2 = 2p^2/r^4$ diverges at the $r = 0$ core.
- Developed a formalism to perform electromagnetic duality on $f(F^2)$
  - Obtained regular electrically-charged solutions with finite F^2 everywhere.
  - Obtained two new explicit examples of self-dual $f(F^2)$ theories.
- Studied properties of repulson stars and black holes.

Summary

– Einstein theories and equations of motion
– Exact solutions, classifications and techniques
– Global analysis: black rings
– Euclidean signature: gravitational instantons.
– Some general properties by energy conditions
– Regular black holes