Minimally Modified Gravity with Auxiliary Constraints Formalism

Zhi-Bang Yao
zhbyao@gmail.com

School of Physics and Astronomy
Sun Yat-sen University

May 17, 2023
APSW-GC@Hangzhou

Based on arXiv: 2302.02090 [gr-qc] accepted by PRD
Collaborated with Michele Oliosi, Xian Gao and Shinji Mukohyama

Outline

Minimally Modified Gravity:

Minimally Modified Gravity (MMG) is a kind of special MG theories that propagates only Two Tensorial DOF (TTDOF) [1] without arising extra DOF motivated by:

- phenomenally, to propose the candidates corresponding to the tensor polarizations signals from GWs events;
- theoretically, to investigate the possibilities of the existence of MMG theories;
- additionally, to provide some insights to the problems of cosmology.

[1] The "TTDOF" also refers to the Transverse and Traceless DOF.

Backgroud

Attempt on MMG

$D \to 4$ Einstein-Gauss-Bonnet gravity [1]:

GR with a dimensionally regularized GB term

$$S_{\text{EGB}}^{\text{4D}} = \lim_{D\to 4} \frac{1}{2\kappa^2} \int \mathrm{d}^D x \sqrt{-g} \left(R - 2\Lambda + \frac{\alpha}{D-4} R_{\text{GB}}^2 \right) \tag{1}$$

is a MG theory with TTDOF bypassing the Lovelock theorem?

Lovelock theorem[2]:

GR is the unique gravitational theory propagating TTDOF with the following conditions:

- 4-dim theory with locality;
- Metric filed with 2nd-order EoM;
- General covariance;

[1] [Glavan and Lin, PRL, 2020]

[2] [Lovelock, J. Math. Phys. 1972]

Backgroud

Attempt on MMG

$D \to 4$ Einstein-Gauss-Bonnet gravity [1]:

GR with a dimensionally regularized GB term

$$S_{\text{EGB}}^{\text{4D}} = \lim_{D\to 4} \frac{1}{2\kappa^2} \int \mathrm{d}^D x \sqrt{-g} \left(R - 2\Lambda + \frac{\alpha}{D-4} R_{\text{GB}}^2 \right) \quad (1)$$

is a MG theory with TTDOF bypassing the Lovelock theorem?

Lovelock theorem[2]:

GR is the unique gravitational theory propagating TTDOF with the following conditions:

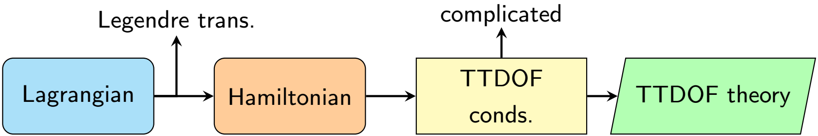
- 4-dim theory with locality;
- Metric filed with 2nd-order EoM;
- General covariance; $\to$ Spatial covariance.

[1] [Glavan and Lin, PRL, 2020]
[2] [Lovelock, J. Math. Phys. 1972]

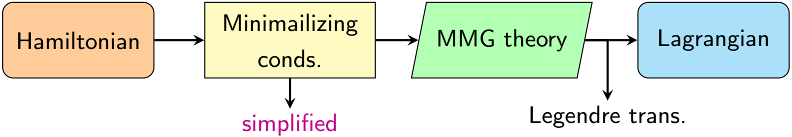
Background

Two approaches to MMG

The Lagrangian approach[1] [2]:



The Hamiltonian approach[3] [4]:



[1] [C. Lin and S. Mukohyama, JCAP, 2017]
[2] [X. Gao and Z.-B. Yao, PRD, 2020]
[3] [S. Mukohyama and K. Noui, JCAP, 2019]
[4] [Z.-B. Yao, M. Oliosi, X. Gao and S. Mukohyama, PRD, 2021]

MMG with ACs: the formalism

A naive framework with spatial covariance:

To a general total Hamiltonian w.r.t. spatial diffeomorphism

$$H_{\mathrm{T}} = \int \mathrm{d}^3x \Big[\underbrace{\mathscr{H}\left(N, \pi, h_{ij}, \pi^{ij}; \nabla_i\right)}_{\text{arbitary function}} + \underbrace{N^i \mathcal{H}_i + \lambda^i \pi_i}_{\text{3d-diff constr.}} \Big] \qquad (2)$$

we can count the number of DOF by

$$\Gamma : \underbrace{\left(\overbrace{N, N^i, h_{ij}}^{\text{ADM-var}} ; \overbrace{\pi, \pi_i, \pi^{ij}}^{\text{Conj-mmta}} \right)}_{\text{20-dim phase space}} \begin{cases} 4_{\mathrm{s}} \\ 4_{\mathrm{v}} \\ 2_{\mathrm{t}} \end{cases} + \underbrace{\mathcal{H}_i \approx 0_i, \pi_i \approx 0_i}_{\text{3d-diff., 1st-class}} \begin{cases} \rightarrow 2_{\mathrm{s}} \\ \rightarrow 0_{\mathrm{v}} \\ \rightarrow 2_{\mathrm{t}} \end{cases}$$

MMG with ACs: the formalism

Constructing the framework

A naive framework with spatial covariance:

To a general total Hamiltonian w.r.t. spatial diffeomorphism

$$H_{\mathrm{T}} = \int \mathrm{d}^3 x \Big[\underbrace{\mathscr{H}\left(N, \pi, h_{ij}, \pi^{ij}; \nabla_i\right)}_{\text{arbitary function}} + \underbrace{N^i \mathcal{H}_i + \lambda^i \pi_i}_{\text{3d-diff constr.}} \Big] \qquad (2)$$

we can count the number of DOF by

$$\Gamma : \underbrace{\left(\overbrace{N, N^i, h_{ij}}^{\text{ADM-var}}; \overbrace{\pi, \pi_i, \pi^{ij}}^{\text{Conj-mmta}} \right)}_{\text{20-dim phase space}} \begin{cases} 4_{\mathrm{s}} \\ 4_{\mathrm{v}} \\ 2_{\mathrm{t}} \end{cases} + \underbrace{\mathcal{H}_i \approx 0_i, \pi_i \approx 0_i}_{\text{3d-diff., 1st-class}} \begin{cases} \to 2_{\mathrm{s}} \to 0_{\mathrm{s}} \\ \to 0_{\mathrm{v}} \\ \to 2_{\mathrm{t}} \end{cases}$$

MMG

MMG with ACs: the formalism

Constructing of framework

A consistent framework with ACs:

To a general total Hamiltonian with auxiliary constraints

$$H_{\mathrm{T}} = \int \mathrm{d}^3x \Big(\underbrace{\mathscr{H}}_{\text{free-fun.}} + \underbrace{N^i \mathcal{H}_i + \lambda^i \pi_i}_{\text{3-diff.}} + \mu_{\mathrm{n}} \underbrace{\mathcal{S}^{\mathrm{n}}}_{\text{ACs}} \Big), \tag{3}$$

with $\mathrm{n} = 1, \cdots, \mathcal{N} (\mathcal{N} \leq 4)$ and we count the number of DOF by

$$\underbrace{\left(\Gamma + \overbrace{\mathcal{H}_i, \pi_i}^{\text{3d-diff}} \right)}_{\text{constrained phase space}} \begin{cases} 2_{\mathrm{s}} \\ 0_{\mathrm{v}} \\ 2_{\mathrm{t}} \end{cases} + \underbrace{\overbrace{\mathcal{S}^{\mathrm{n}}}^{\text{prim.}} + \overbrace{\dot{\mathcal{S}}^{\mathrm{n}}}^{\text{seco.}} + \cdots}_{\#^{\mathrm{s}}_{\mathrm{1st}} + \#^{\mathrm{s}}_{\mathrm{2nd}}} \begin{cases} \to 2_{\mathrm{s}} - \#^{\mathrm{s}}_{\mathrm{1st}} - \frac{1}{2}\#^{\mathrm{s}}_{\mathrm{2nd}} \\ \to 0_{\mathrm{v}} \\ \to 2_{\mathrm{t}} \end{cases}$$

For the purpose of constructing a MMG theory, we require

$$2_{\mathrm{s}} - \#^{\mathrm{s}}_{\mathrm{1st}} - \frac{1}{2}\#^{\mathrm{s}}_{\mathrm{2nd}} = 0, \quad 0 \leq \#^{\mathrm{s}}_{\mathrm{1st}} + \#^{\mathrm{s}}_{\mathrm{2nd}} \leq 4. \tag{4}$$

MMG with ACs: the formalism

Results collections

# ACs	Minimalizing cond.	Symmetrizing cond.	Classifications	Ident. key	Examples
$\#^{\mathrm{s}} = 4$	none	none	$\#^{\mathrm{s}}_{\mathrm{1st}} = 0,\ \#^{\mathrm{s}}_{\mathrm{2nd}} = 4$	IV-0-4	Mixed Traces
$\#^{\mathrm{s}} = 3$	$\left[S^1, S^{\mathrm{n}}\right]$	$\left[S^1, \mathscr{H}\right]$	$\#^{\mathrm{s}}_{\mathrm{1st}} = 1,\ \#^{\mathrm{s}}_{\mathrm{2nd}} = 2$	III-1-2	unknown
		none	$\#^{\mathrm{s}}_{\mathrm{1st}} = 0,\ \#^{\mathrm{s}}_{\mathrm{2nd}} = 4$	III-0-4	unknown
$\#^{\mathrm{s}} = 2$	$\left[S^1, S^{\mathrm{n}}\right] \& \left[S^2, S^2\right]$	$\left[S^1, \mathscr{H}\right]\ \&\ \left[S^2, \mathscr{H}\right]$	$\#^{\mathrm{s}}_{\mathrm{1st}} = 2,\ \#^{\mathrm{s}}_{\mathrm{2nd}} = 0$	II-2-0	unknown
		$\left[S^2, \dot{S}^1\right]\ \&\ \left[S^2, \mathscr{H}\right]$	$\#^{\mathrm{s}}_{\mathrm{1st}} = 1,\ \#^{\mathrm{s}}_{\mathrm{2nd}} = 2$	II-1-2b	unknown
		none	$\#^{\mathrm{s}}_{\mathrm{1st}} = 0,\ \#^{\mathrm{s}}_{\mathrm{2nd}} = 4$	II-0-4b	Linear AC
	$\left[S^1, S^{\mathrm{n}}\right]\ \&\ \left[S^1, \dot{S}^1\right]$	$\left[\dot{S}^1, H_{\mathrm{P}}\right]$	$\#^{\mathrm{s}}_{\mathrm{1st}} = 1,\ \#^{\mathrm{s}}_{\mathrm{2nd}} = 2$	II-1-2a	4dEGB
		none	$\#^{\mathrm{s}}_{\mathrm{1st}} = 0,\ \#^{\mathrm{s}}_{\mathrm{2nd}} = 4$	II-0-4a	unknown
$\#^{\mathrm{s}} = 1$	$\left[S^1, S^1\right], \left[S^1, \dot{S}^1\right]\ \&\ \left[\dot{S}^1, \dot{S}^1\right]$	$\left[\dot{S}^1, \mathscr{H}\right]$	$\#^{\mathrm{s}}_{\mathrm{1st}} = 2,\ \#^{\mathrm{s}}_{\mathrm{2nd}} = 0$	I-2-0	GR & $f\left(\mathscr{H}\right)$
		$\left[\dot{S}^1, \ddot{S}^1\right]$	$\#^{\mathrm{s}}_{\mathrm{1st}} = 1,\ \#^{\mathrm{s}}_{\mathrm{2nd}} = 2$	I-1-2b	unknown
	$\left[S^1, S^1\right], \left[S^1, \dot{S}^1\right]\ \&\ \left[S^1, \ddot{S}^1\right]$	$\left[\ddot{S}^1, \mathscr{H}\right]$	$\#^{\mathrm{s}}_{\mathrm{1st}} = 1,\ \#^{\mathrm{s}}_{\mathrm{2nd}} = 2$	I-1-2a	Cuscuton & QEC
		none	$\#^{\mathrm{s}}_{\mathrm{1st}} = 0,\ \#^{\mathrm{s}}_{\mathrm{2nd}} = 4$	I-0-4	unknown

Table 1: The minimalizing and symmetrizing conditions.

Note that we simply denote the condition $\left[\cdot\left(\vec{x}\right), \cdot\left(\vec{y}\right)\right] \approx 0$ by $[\cdot, \cdot]$ in the table.

MMG with ACs: the examples

A special case of I-2-0 type of MMG

General Relativity:

The total Hamiltonian of GR

$$H_{\mathrm{T}}^{(\mathrm{GR})} = \int \mathrm{d}^3 x \Big(\underbrace{N \overbrace{\mathcal{H}_0^{(\mathrm{GR})}}^{\dot{\mathcal{S}}^1} + \lambda \overbrace{\pi}^{\mathcal{S}^1}}_{\text{time-repar.}} + \underbrace{N^i \mathcal{H}_i + \lambda^i \pi_i}_{\text{3d-diff.}} \Big), \qquad (5)$$

with the momentum constraint

$$\mathcal{H}_0^{(\mathrm{GR})} \equiv \frac{1}{\sqrt{h}} \left(\pi^{ij} \pi_{ij} - \frac{1}{2} \pi^i_i \pi^j_j \right) - \sqrt{h} \left(R + \Lambda \right). \qquad (6)$$

Cayley-Hamilton construction with mixed traces constraints

We construct a concrete model of MMG with four ACs which can be used to couple with matter consistently

$$H_{\mathrm{T}}^{(\mathrm{C.H.})} = \int \mathrm{d}^3x \Big[\mathscr{H}^{(\mathrm{C.H.})} + N^i \mathcal{H}_i + \lambda^i \pi_i + \lambda \underbrace{\pi}_{\mathcal{S}^4} + \mu_{\mathrm{I}} \Big(\underbrace{\mathscr{Q}^{\mathrm{I}} - \mathscr{P}^{\mathrm{I}}(N)}_{\mathcal{S}^1 \sim \mathcal{S}^3} \Big) \Big], \tag{7}$$

where we choose the mixed traces Q^{I} as three ACs

$$R^{\mathrm{I}} \equiv \left\{ R^i_i, R^i_j R^j_i, R^i_j R^j_k R^k_i \right\}, \tag{8}$$

$$\Pi^{\mathrm{I}} \equiv \left\{ \pi^i_i, \pi^i_j \pi^j_i, \pi^i_j \pi^j_k \pi^k_i \right\}, \tag{9}$$

$$Q^{\mathrm{I}} \equiv \left\{ R^i_j \pi^j_i, R^i_j \pi^j_k \pi^k_i, R^i_j R^j_k \pi^k_i \right\}. \tag{10}$$

MMG with ACs: the examples

A concrete model with four ACs

Tensor perturbations:

On the flat FLRW background $h_{ij} = a(t)^2 \mathfrak{g}_{ij}$ with

$$\mathfrak{g}_{ij} \equiv \delta_{ij} + \gamma_{ij} + \frac{1}{2!}\gamma_{ik}\gamma^{k}{}_{j} + \frac{1}{3!}\gamma_{ik}\gamma^{k}{}_{l}\gamma^{l}{}_{j} + \cdots \,, \tag{11}$$

we derive the quadratic action as follows

$$S_2^{(\mathrm{C.H.})} = \int \mathrm{d}t\mathrm{d}^3x \frac{1}{4}\Big(G_0(t)\,\dot{\gamma}_{ij}\dot{\gamma}^{ij} + W_0(t)\,\gamma_{ij}\frac{\Delta}{a^2}\gamma^{ij} - W_2(t)\,\gamma_{ij}\frac{\Delta^2}{a^4}\gamma^{ij}\Big),$$

where

$$G_0(t) \equiv \left[\left(\frac{\partial \bar{\mathscr{H}}}{\partial \Pi^2}\right)^2 - 3\frac{\partial \bar{\mathscr{H}}}{\partial \Pi^3}\left(\frac{\partial \bar{\mathscr{H}}}{\partial \Pi^1} - 2H\right)\right]^{-1/2}, \tag{12}$$

$$W_0(t) \equiv -\frac{\partial \bar{\mathscr{H}}}{\partial \mathrm{R}^1} + \varpi_0(t)\,, \quad W_2(t) \equiv \frac{\partial \bar{\mathscr{H}}}{\partial \mathrm{R}^2} + \varpi_2(t)\,. \tag{13}$$

MMG with ACs: the examples

A concrete model with four ACs

Dispersion relation:

$$\begin{aligned}\omega_{\mathsf{T}}^{2} &= \frac{\mathsf{W}_0(\tau)}{G_0(\tau)}\frac{k^2}{a^2}+\frac{\mathsf{W}_2(\tau)}{G_0(\tau)}\frac{k^4}{a^4}\\ &= \frac{k^2}{a^2}G_0^{-1}\left[\varpi_0-\frac{\partial\bar{\mathscr{H}}}{\partial\mathsf{R}^1}+\left(\varpi_2+\frac{\partial\bar{\mathscr{H}}}{\partial\mathsf{R}^2}\right)\frac{k^2}{a^2}\right].\end{aligned} \tag{14}$$

On large scales, the speed of GWs $c_{\mathsf{T}}=\omega_{\mathsf{T}}/k=1$ when

$$\frac{\partial\bar{\mathscr{H}}}{\partial\mathsf{R}^1}=\varpi_0-G_0, \tag{15}$$

which, as well as the modified dispersion relation, should be constrained by

$$-3\times10^{-15}<\frac{\mathsf{W}_0}{G_0}-1<7\times10^{-16},\quad \left|\frac{\mathsf{W}_2}{G_0}\right|<10^{-19}\ \mathrm{peV}^{-2}. \tag{16}$$

Summary

- Put forward the concept of "auxiliary constraint" and developed it into a formalism;
- Found out all possible constraints structures for the MMG theories with ACs and derived the corresponding minimalizing and symmetrizing conditions for each cases;
- Constructed a concrete MMG model with four ACs and derived the dispersion relation for the gravitational waves.

Summary

- Put forward the concept of "auxiliary constraint" and developed it into a formalism;
- Found out all possible constraints structures for the MMG theories with ACs and derived the corresponding minimalizing and symmetrizing conditions for each cases;
- Constructed a concrete MMG model with four ACs and derived the dispersion relation for the gravitational waves.

Thank you for your attention!

Background

Case of $\mathcal{N} = 3$

The partial Dirac matrix with three ACs:

$[\cdot(\vec{x}), \cdot(\vec{y})]$	$\mathcal{S}^1$	$\mathcal{S}^2$	$\mathcal{S}^3$
$\mathcal{S}^1$	$\times$	$\times$	$\times$
$\mathcal{S}^2$	$\times$	$\times$	$\times$
$\mathcal{S}^3$	$\times$	$\times$	$\times$

Classification: $\#^{\mathsf{s}}_{\mathrm{1st}} = 1, \#^{\mathsf{s}}_{\mathrm{2nd}} = 2$

$[\cdot(\vec{x}), \cdot(\vec{y})]$	$\mathcal{S}^1$	$\mathcal{S}^2$	$\mathcal{S}^3$
$\mathcal{S}^1$	0	0	0
$\mathcal{S}^2$	0	$\times$	$\times$
$\mathcal{S}^3$	0	$\times$	$\times$

According to the time evolution of the ACs, we require

$$\dot{\mathcal{S}}^1(\vec{x}) = \int \mathrm{d}^3 y \Big\{ \underbrace{\left[\mathcal{S}^1(\vec{x}), \mathcal{S}^{\mathsf{n}}(\vec{y})\right]}_{\text{minimalizing cond.}} \mu_{\mathsf{n}}(\vec{y}) + \underbrace{\left[\mathcal{S}^1(\vec{x}), \mathscr{H}(\vec{y})\right]}_{\text{symmetrizing cond.}} \Big\} \approx 0.$$

Background

A special case of I-1-2a type of MMG

The Cuscuton [1]:

The total Hamiltonian of Cuscuton

$$H_{\mathrm{T}}^{(\mathrm{Cus})} = \int \mathrm{d}^3x \left(N \underbrace{\mathcal{H}_0^{(\mathrm{Cus})}}_{\text{2nd}} + \lambda \underbrace{\pi}_{\text{1st}} + \underbrace{N^i \mathcal{H}_i + \lambda^i \pi_i}_{\text{3d-diff.}} \right), \tag{17}$$

with the momentum constraint

$$\mathcal{H}_0^{(\mathrm{Cus})} \equiv \frac{1}{\sqrt{h}} \left(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^i_i \pi^j_j \right) - \sqrt{h} \left(R + V + \frac{\mu^2}{N} \right). \tag{18}$$

[1] [N. Afshordi, D. J. H. Chung and G. Geshnizjani, PRD, 2007]

By performing the Legendre transformation, we formally obtain the corresponding action as follows

$$S = \int \mathrm{d}t\mathrm{d}^3x \left[N \left(\pi F + 2\pi^{ij} K_{ij} \right) - \mathscr{H} - \mu_{\mathsf{n}} \mathcal{S}^{\mathsf{n}} \right], \tag{19}$$

where π and π^{ij} should be understood as the solutions of

$$NF = \frac{\delta H_{\mathsf{P}}}{\delta \pi}, \quad 2NK_{ij} = \frac{\delta H_{\mathsf{P}}}{\delta \pi^{ij}} \tag{20}$$

with

$$F \equiv \frac{1}{N} \left(\dot{N} - N^i \nabla_i N \right), \quad K_{ij} \equiv \frac{1}{2N} \left(\dot{h}_{ij} - 2\nabla_{(i} N_{j)} \right). \tag{21}$$

Backup

The coefficients

$$\begin{aligned}
\varpi_0\,(t) \quad \equiv \quad & -\frac{1}{2}\mathcal{G}_0\bar{\mu}_1' + \left(3\frac{\partial\bar{\mathscr{H}}}{\partial\Pi^3}\right)^{-1}\left(\mathcal{G}_0\frac{\partial\bar{\mathscr{H}}}{\partial\Pi^2} - 1\right)\bar{\mu}_2' \\
& + \left[\left(3\frac{\partial\bar{\mathscr{H}}}{\partial\Pi^3}\right)^{-1}\left(\frac{\partial\bar{\mathscr{H}}}{\partial\Pi^2} - \mathcal{G}_0^{-1}\right) - \frac{\mathcal{G}_0'}{2} + \mathcal{G}_0 H\right]\bar{\mu}_1 \\
& + \left(3\frac{\partial\bar{\mathscr{H}}}{\partial\Pi^3}\mathcal{G}_0\right)^{-2}\Bigg[-1 + \mathcal{G}_0\Bigg(3\mathcal{G}_0(\mathcal{G}_0\frac{\partial\bar{\mathscr{H}}}{\partial\Pi^3}\frac{\partial\bar{\mathscr{H}}}{\partial\Pi^2}' \\
& + \frac{\partial\bar{\mathscr{H}}}{\partial\Pi^3}' + 2\frac{\partial\bar{\mathscr{H}}}{\partial\Pi^3}H) + \frac{\partial\bar{\mathscr{H}}}{\partial\Pi^2}(2 - 3\mathcal{G}_0(\mathcal{G}_0\frac{\partial\bar{\mathscr{H}}}{\partial\Pi^3}' \\
& - \frac{\partial\bar{\mathscr{H}}}{\partial\Pi^3}\mathcal{G}_0' + 2\frac{\partial\bar{\mathscr{H}}}{\partial\Pi^3}\mathcal{G}_0 H)) - \frac{\partial\bar{\mathscr{H}}}{\partial\Pi^2}^2\mathcal{G}_0\Bigg)\Bigg]\bar{\mu}_2. \qquad (22)
\end{aligned}$$

Backup

The coefficients

$$\varpi_2(t) \equiv \left(6\frac{\partial\bar{\mathscr{H}}}{\partial\Pi^3}\right)^{-2}\left[12\frac{\partial\bar{\mathscr{H}}}{\partial\Pi^3}\left(\mathcal{G}_0^{-1}-\frac{\partial\bar{\mathscr{H}}}{\partial\Pi^2}\right)\bar{\mu}_3 \right.$$
$$\left. -\mathcal{G}_0\left(3\frac{\partial\bar{\mathscr{H}}}{\partial\Pi^3}\bar{\mu}_1+2\left(\mathcal{G}_0^{-1}-\frac{\partial\bar{\mathscr{H}}}{\partial\Pi^2}\right)\bar{\mu}_2\right)^2\right]. \quad (23)$$