BLACK HOLE CHEMISTRY

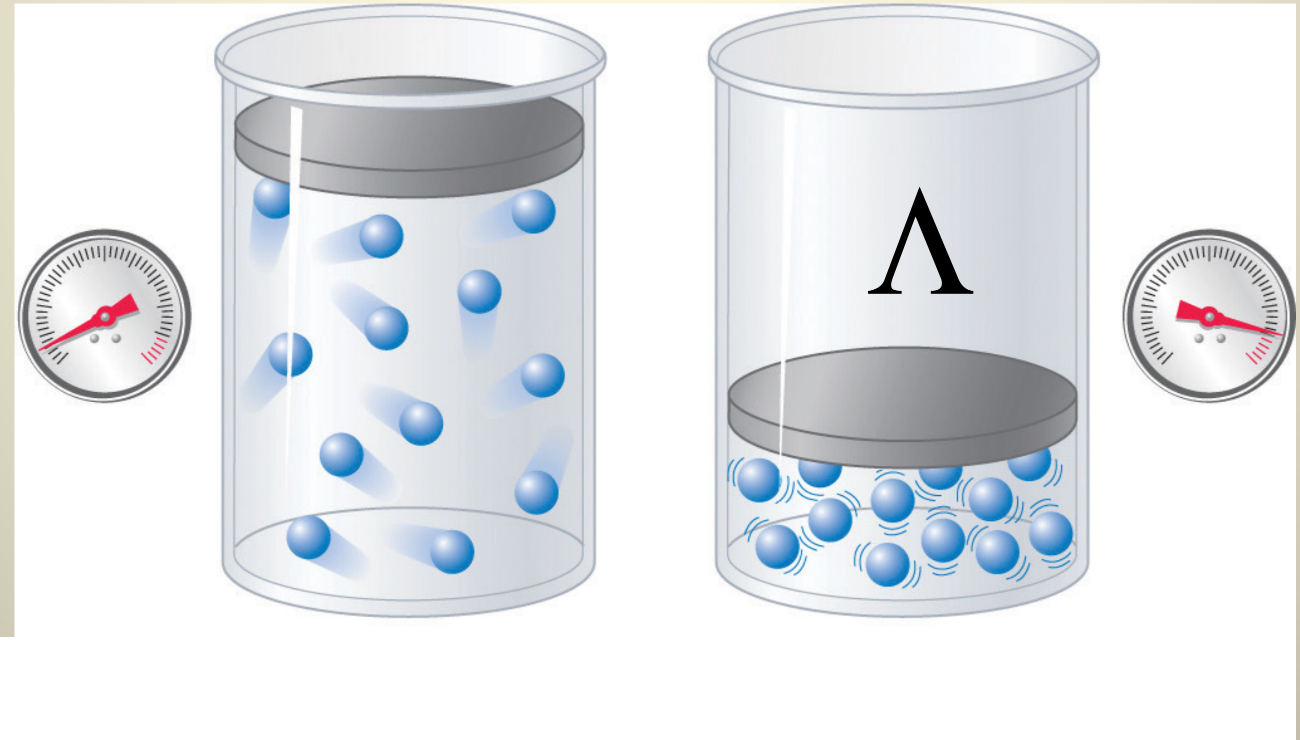
Robert Mann

University of Waterloo
Waterloo, Canada

https://boardgamegeek.com/image/2241156/alchemists

Black Hole Chemistry: Pressure from the Vacuum

$$p = -\frac{\Lambda}{8\pi G}$$



Recent Results in Black Hole Chemistry

- Accelerating Black Hole Thermodynamics
- Holographic Black Hole Chemistry
  - Holographic Smarr Relation
  - Holographic Complexity and Thermodynamic Volume
  - Central Charge Criticality
- Black Hole Multicriticality
- And more!

Accelerating Black Holes

- Basic features
  - both an event horizon and an acceleration horizon
  - Thermodynamic equilibrium can't be maintained
- Asymptotically AdS
  - Acceleration horizon removed for small acceleration
  - Cosmic string suspends BH away from the centre of AdS
- Conflicting Results for Thermodynamics
  - Differing identifications of mass
  - Role of conical deficits in first law not clear
  - Free energy/action not compatible

Appels/Gregory/ Kubiznak PRL **117** (2016) 131303;
JHEP **1705** (2017)116
Astorino PRD**95** (2017) 064007
Anabalon/Appels/Gregory/
Kubiznak/Mann/Ovgun PRD

Metric for ABH in AdS

$$ds^2 = \frac{1}{\Omega^2}\left[-f dt^2 + \frac{dr^2}{f} + r^2\left(\frac{d\theta^2}{\Sigma} + \Sigma \sin^2\theta \frac{d\phi^2}{K^2}\right)\right]$$

$$\Omega = 1 + Ar\cos\theta \qquad \Sigma = 1 + 2mA\cos\theta$$

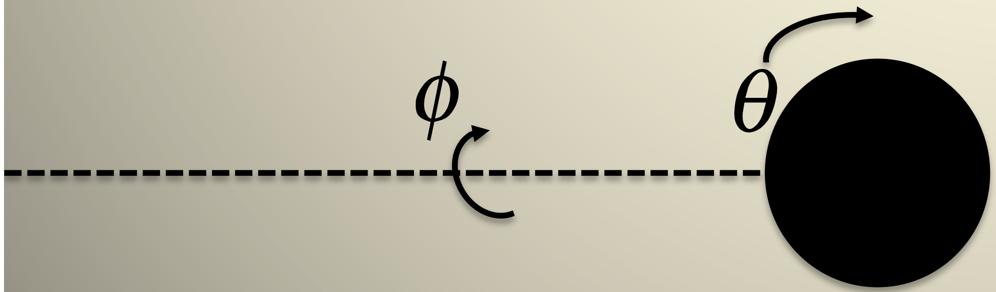
Hong/Teo
CQG **20** (2003) 3269

$$f(r) = (1 - A^2 r^2)\left(1 - \frac{2m}{r}\right) + \frac{r^2}{\ell^2}$$

String
Tension

$$\mu_\pm = \frac{\delta_\pm}{8\pi} = \frac{1}{4}\left(1 - \frac{\Sigma(\theta_\pm)}{K}\right) = \frac{1}{4}\left(1 - \frac{1 \pm 2mA}{K}\right)$$

Parameterizes
deficit angle



Metric for ABH in AdS

$$ds^2 = \frac{1}{\Omega^2}\Big[-f dt^2 + \frac{dr^2}{f} + r^2\Big(\frac{d\theta^2}{\Sigma} + \Sigma \sin^2\theta \frac{d\phi^2}{K^2}\Big)\Big]$$

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$$\Omega = 1 + Ar\cos\theta$$

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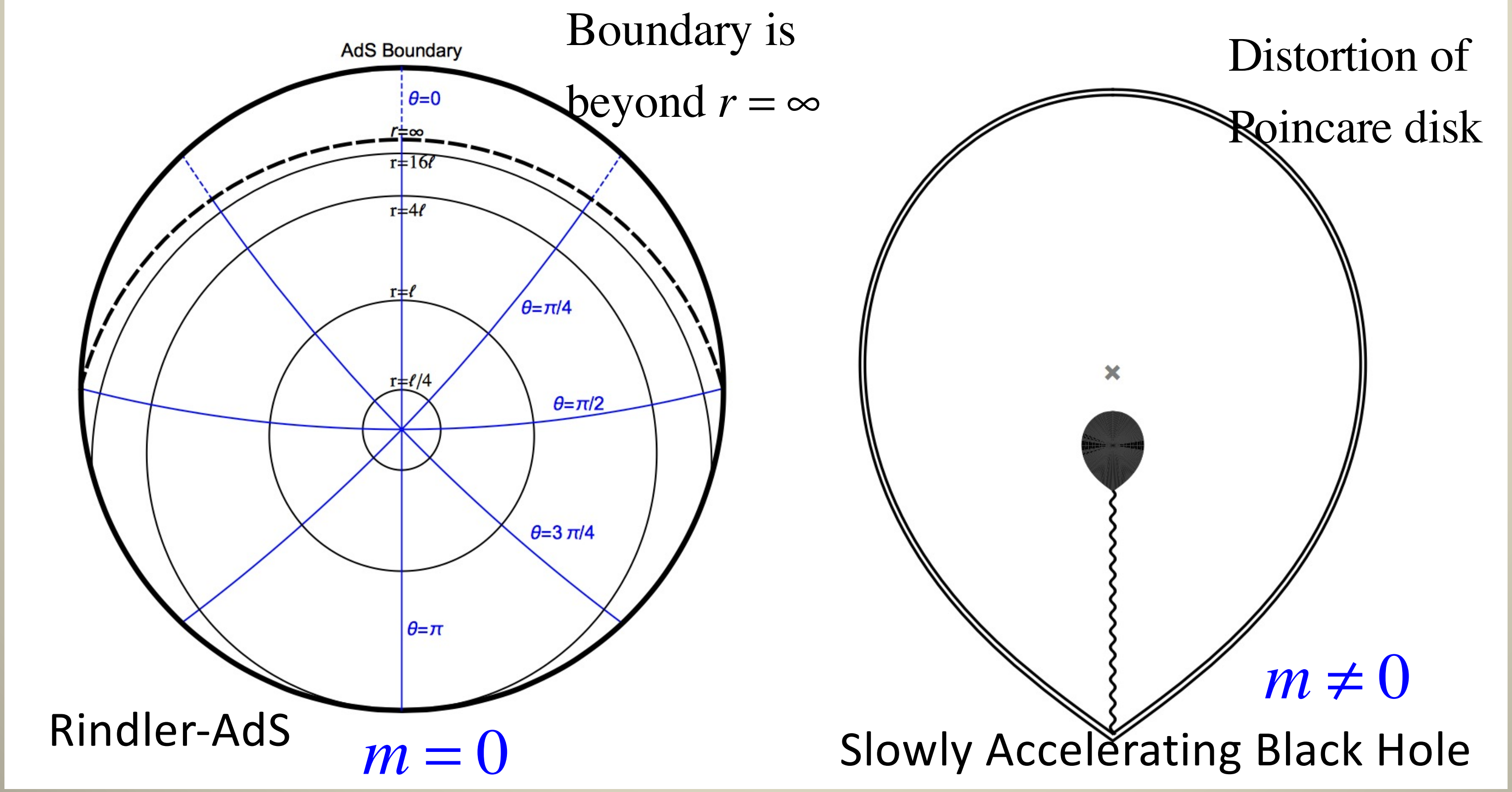


Boundary is beyond $r = \infty$

Distortion of Poincare disk

Rindler-AdS $m = 0$

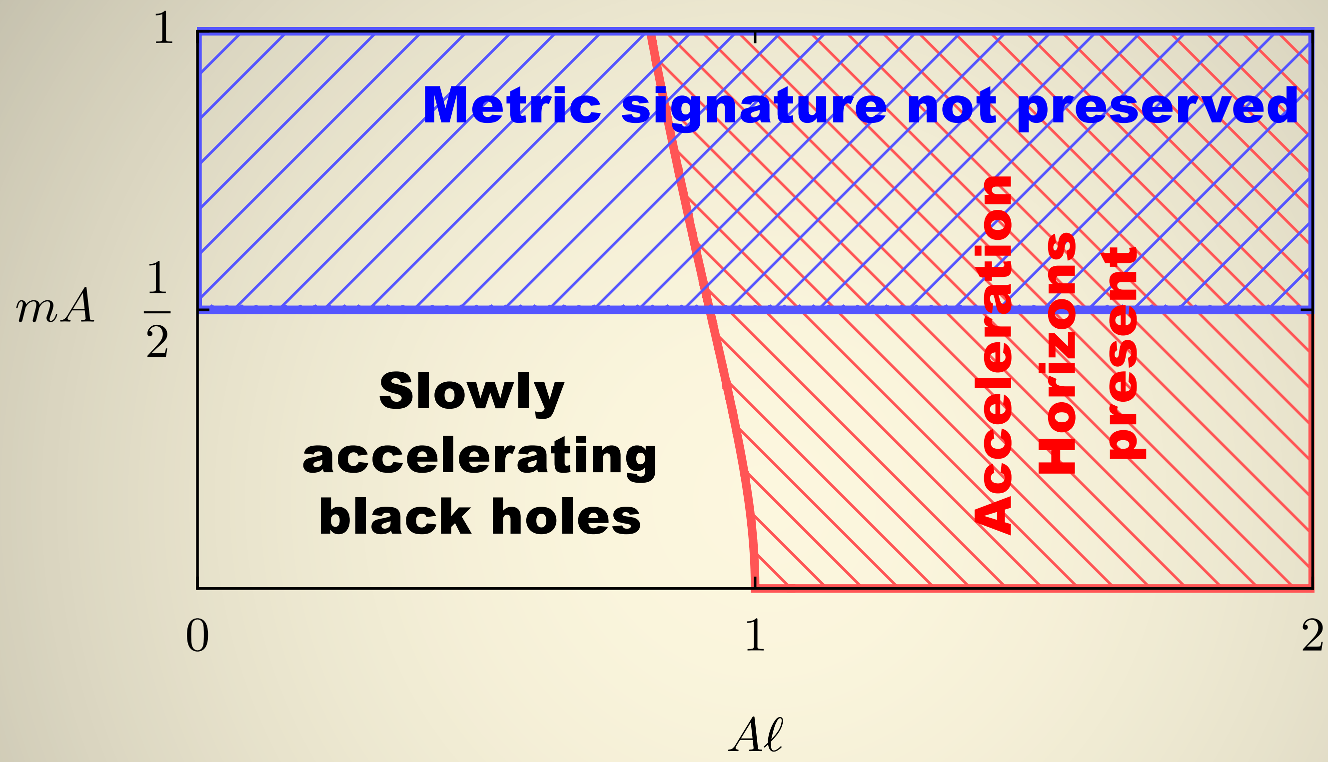
$m \neq 0$

Slowly Accelerating Black Hole

$$A\ell < \frac{3\sqrt{6}}{8}$$

No Acceleration Horizons

$f(-1/A\cos\theta)>1$



$$ds^2=\frac{1}{\Omega^2}\left[-fdt^2+\frac{dr^2}{f}+r^2\left(\frac{d\theta^2}{\Sigma}+\Sigma\sin^2\theta\frac{d\phi^2}{K^2}\right)\right]$$

$$f(r)=(1-A^2r^2)\left(1-\frac{2m}{r}\right)+\frac{r^2}{\ell^2}$$

$$\Omega=1+Ar\cos\theta$$

$$\Sigma=1+2mA\cos\theta$$

Asymptotics

$\Omega = 1 + Ar\cos\theta$

$\Sigma = 1 + 2mA\cos\theta$

$$ds^2 = \frac{1}{\Omega^2}\Big[-f dt^2 + \frac{dr^2}{f} + r^2\Big(\frac{d\theta^2}{\Sigma} + \Sigma \sin^2\theta \frac{d\phi^2}{K^2}\Big)\Big]$$

$$1 + \frac{R^2}{\ell^2} = \frac{1 + (1 - A^2\ell^2)r^2 / \ell^2}{(1 - A^2\ell^2)\Omega^2} \qquad R\sin\vartheta = \frac{r\sin\theta}{\Omega}$$

$m = 0$

$$ds^2_{AdS} = -\Big(1 + \frac{R^2}{\ell^2}\Big)\alpha^2 dt^2 + \frac{dR^2}{1 + \frac{R^2}{\ell^2}} + R^2\Big(d\vartheta^2 + \sin^2\vartheta \frac{d\phi^2}{K^2}\Big)$$

$$\alpha = \sqrt{1 - A^2\ell^2}$$

Correct time coordinate

is $\tau = \alpha t$

Thermodynamic Quantities

Temperature

$$T = \frac{f'(r_+)}{4\pi\alpha} = \frac{1 + 3\frac{r_+^2}{\ell^2} - A^2 r_+^2 \left(2 + \frac{r_+^2}{\ell^2} - A^2 r_+^2\right)}{4\pi\alpha r_+ (1 - A^2 r_+^2)}$$

Entropy

$$S = \frac{\mathcal{A}}{4} = \frac{\pi r_+^2}{K(1 - A^2 r_+^2)}$$

$$f(r_+) = (1 - A^2 r_+^2)\left(1 - \frac{2m}{r_+}\right) + \frac{r_+^2}{\ell^2} = 0$$

Mass

- Thermodynamic
  - Via the Smarr Relation
- Conformal
  - Via Electric part of Weyl Tensor
- Holographic
  - Via AdS/CFT counterterms

$$\left.\right\} M = \frac{\alpha m}{K} = \sqrt{1 - A^2 \ell^2}\,\frac{m}{K}$$

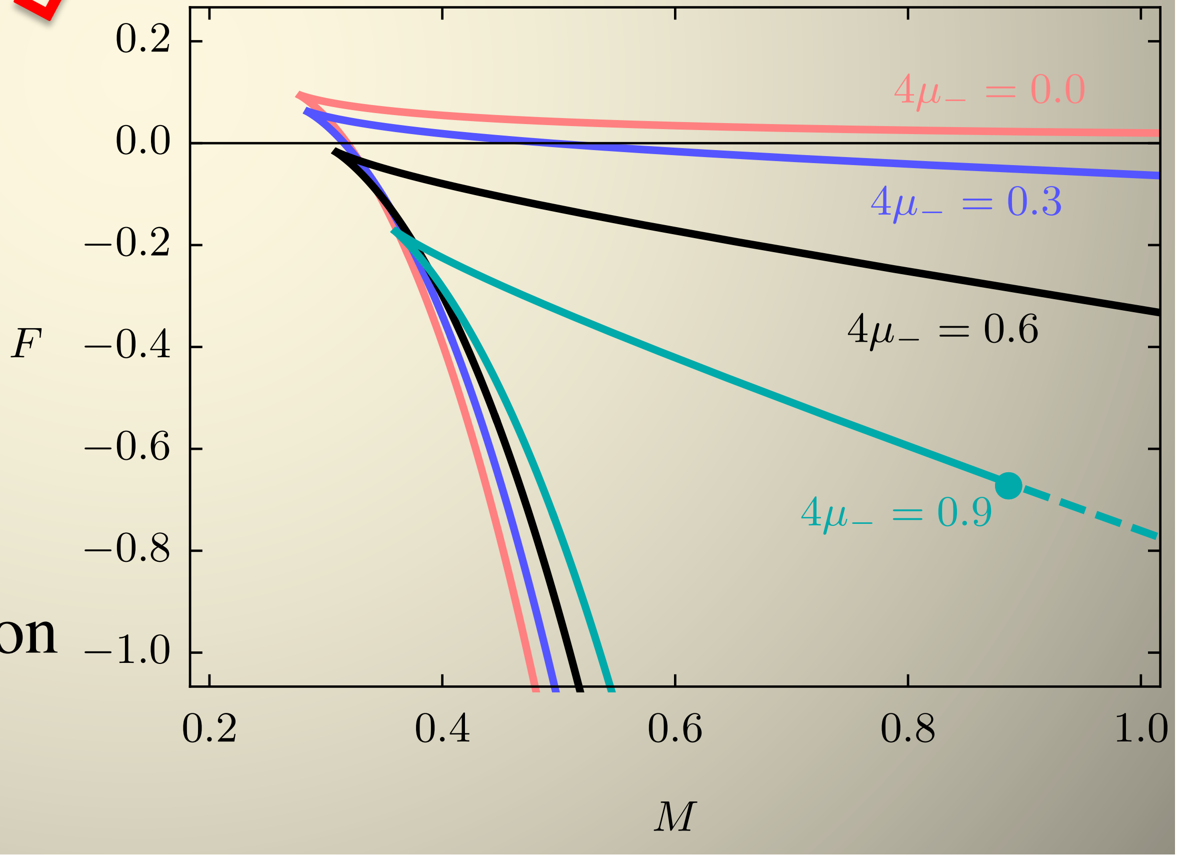
Action

$$I = \frac{\beta}{2\alpha K}\left(m - 2mA^2\ell^2 - \frac{r_+^3}{\ell^2(1-A^2r_+^2)^2} \right)$$

$$\beta = {}^{1}\!/\!_{T}$$

⇨ $F = I/\beta = M - TS$ ✓

$$\mu_\pm = \frac{\delta_\pm}{8\pi} = \frac{1}{4}\left(1 - \frac{1 \pm 2mA}{K}\right)$$



$\mu_- = 0$: Hawking-Page transition

$\mu_- > 0$: No clear interpretation

With Charge and Rotation

Anabalon/Appels/
Gray/Gregory/
Kubiznak/Mann/Ovgun
JHEP 1904 (2019) 096

$$ds^2 = \frac{1}{H^2}\left\{-\frac{f(r)}{\Sigma}\left[\frac{dt}{\alpha} - a\sin^2\theta\frac{d\varphi}{K}\right]^2 + \frac{\Sigma}{f(r)}dr^2 + \frac{\Sigma r^2}{h(\theta)}d\theta^2\right.$$

$$\left. + \frac{h(\theta)\sin^2\theta}{\Sigma r^2}\left[\frac{adt}{\alpha} - (r^2+a^2)\frac{d\varphi}{K}\right]^2\right\}$$

$$\Sigma = 1 + \frac{a^2}{r^2}\cos^2\theta$$

$$H = 1 + Ar\cos\theta$$

$$h(\theta) = 1 + 2mA\cos\theta + \left[A^2(a^2+e^2) - \frac{a^2}{\ell^2}\right]\cos^2\theta$$

$$F = dB \qquad B = -\frac{e}{\Sigma r}\left[\frac{dt}{\alpha} - a\sin^2\theta\frac{d\varphi}{K}\right] + \Phi_t dt$$

$$f(r) = \left(1 - A^2r^2\right)\left[1 - \frac{2m}{r} + \frac{a^2+e^2}{r^2}\right] + \frac{r^2+a^2}{\ell^2}$$

$$M = \frac{m(\Xi + a^2/\ell^2)(1 - A^2\ell^2\Xi)}{K\Xi\alpha(1+a^2A^2)} \qquad T = \frac{f'_+ r_+^2}{4\pi\alpha(r_+^2+a^2)}$$

$$S = \frac{\pi(r_+^2+a^2)}{K(1-A^2r_+^2)} \qquad Q = \frac{e}{K} \qquad \Phi = \Phi_t = \frac{er_+}{(r_+^2+a^2)\alpha}$$

$$\Xi = 1 - \frac{a^2}{\ell^2} + A^2(e^2+a^2)$$

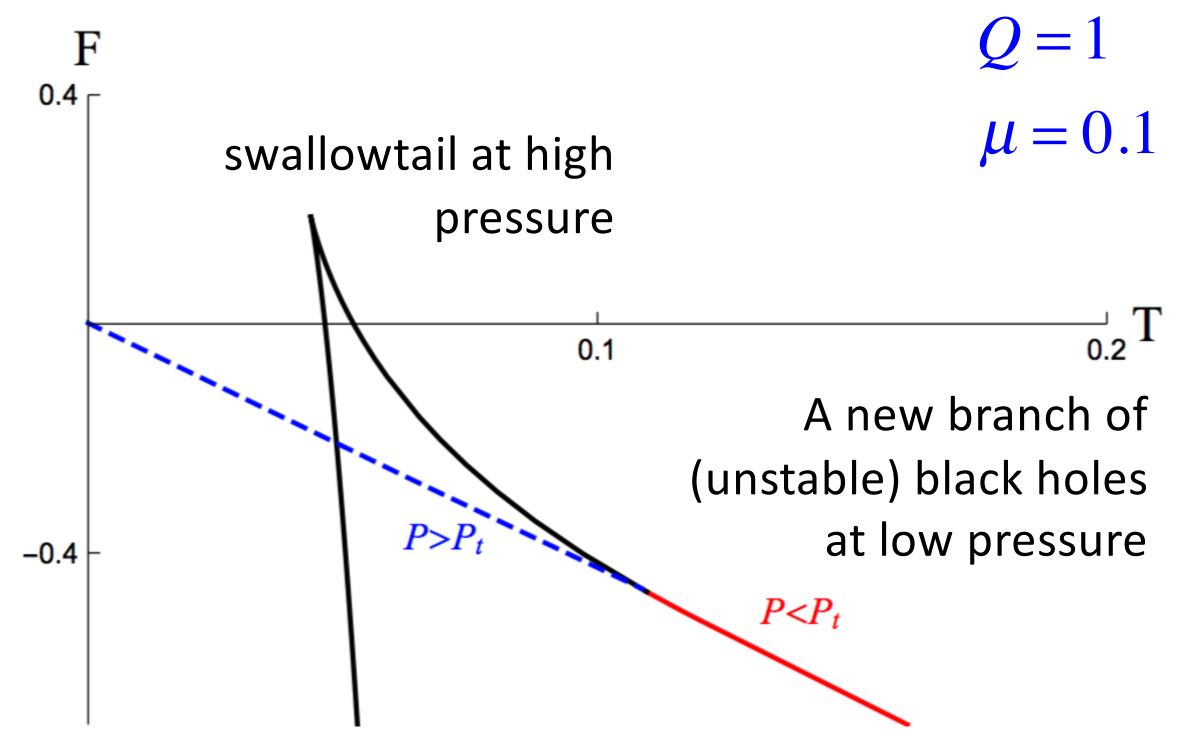
$$J = \frac{ma}{K^2} \qquad \Omega = \Omega_H - \Omega_\infty = \left(\frac{Ka}{\alpha(r_+^2+a^2)}\right) - \left(-\frac{aK(1-A^2\ell^2\Xi)}{\ell^2\Xi\alpha(1+a^2A^2)}\right)$$

$$P = \frac{3}{8\pi\ell^2} \qquad V = \frac{4\pi}{3K\alpha}\left[\frac{r_+(r_+^2+a^2)}{(1-A^2r_+^2)^2} + \frac{m[a^2(1-A^2\ell^2\Xi) + A^2\ell^4\Xi(\Xi + a^2/\ell^2)]}{(1+a^2A^2)\Xi}\right]$$

First law and
Smarr OK!

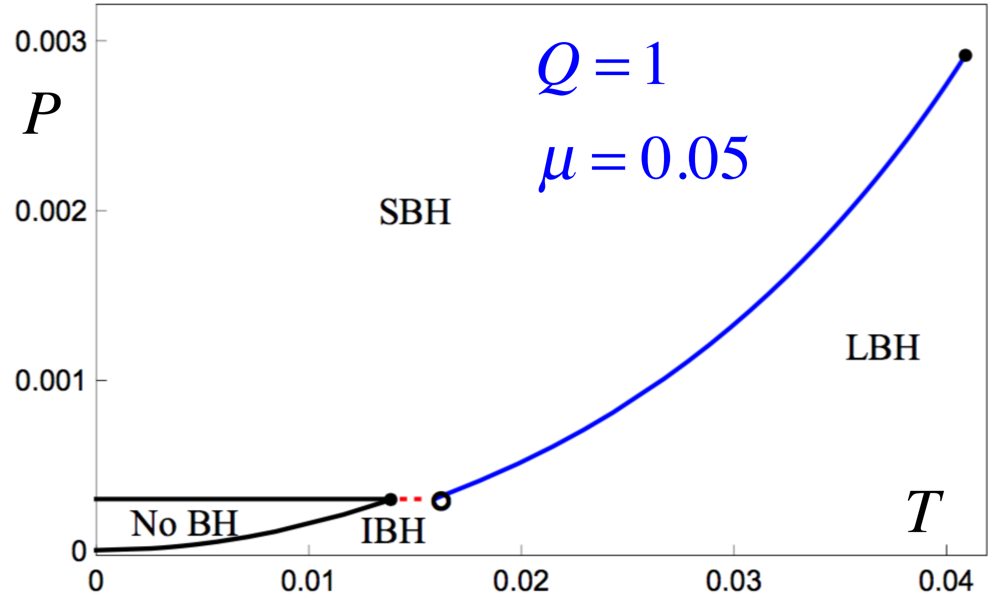
Snapping Swallowtails

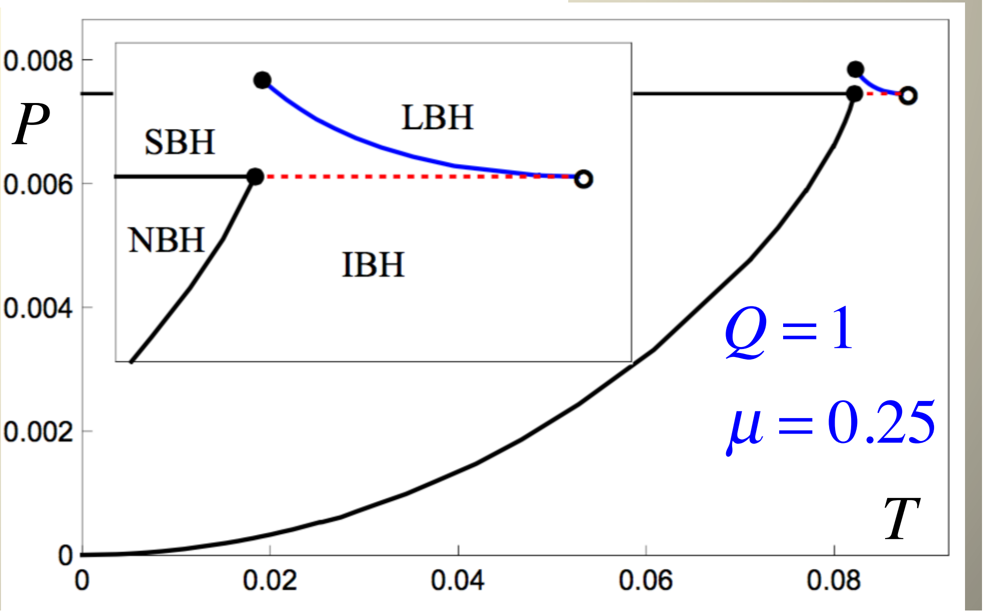
Abbasvandi/Cong
Kubiznak/Mann
CQG 36 (2019) 104001



swallowtail at high pressure

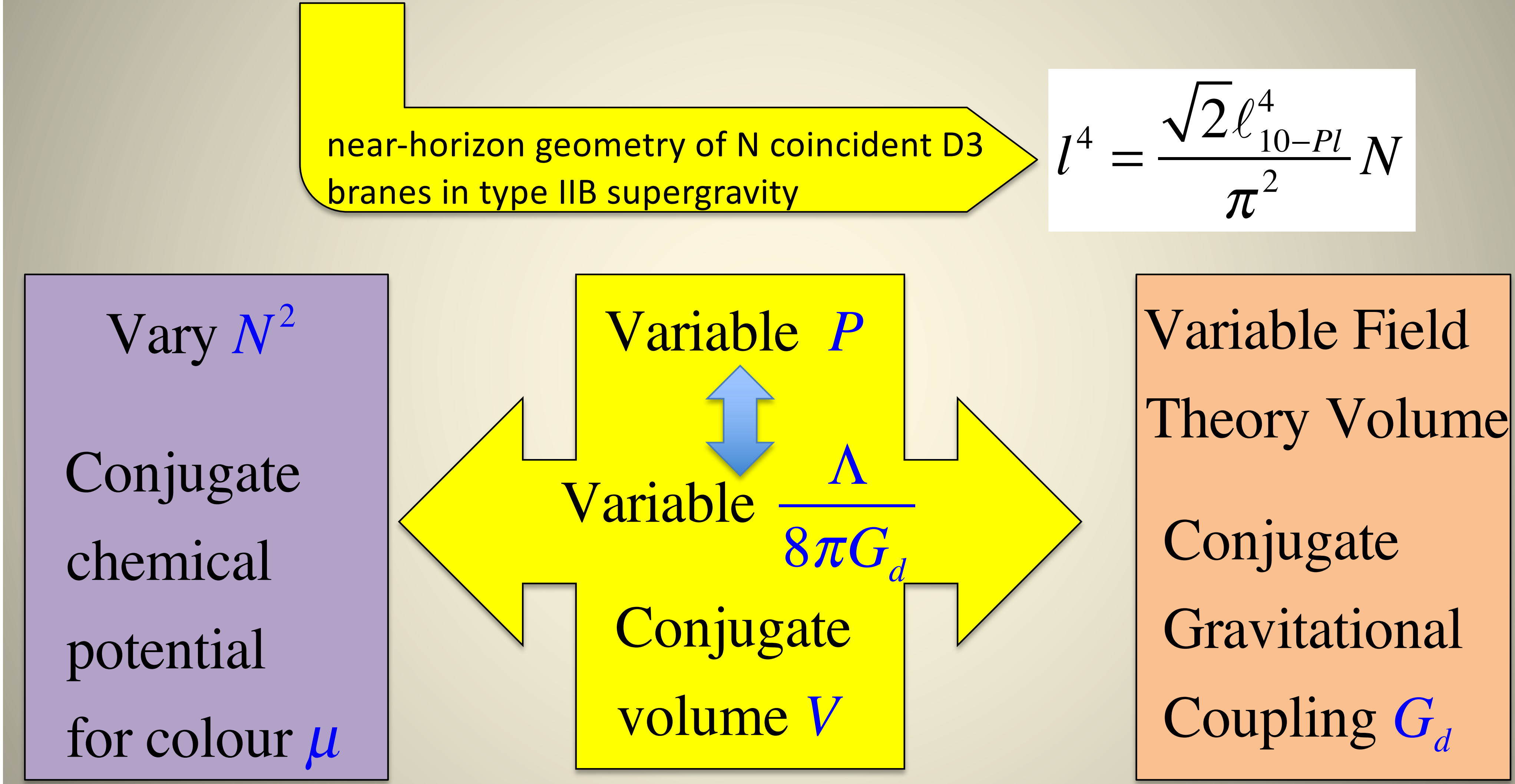
A new branch of (unstable) black holes at low pressure





Variable Λ and AdS/CFT?

$$AdS_5 \times S^5 \Leftrightarrow \mathcal{N} = 4 \text{ SU(N) Super Yang-Mills}$$



near-horizon geometry of N coincident D3 branes in type IIB supergravity

$$l^4 = \frac{\sqrt{2}\,\ell^4_{10-Pl}}{\pi^2} N$$

Vary N^2

Conjugate chemical potential for colour μ

Variable P

Variable $\dfrac{\Lambda}{8\pi G_d}$

Conjugate volume V

Variable Field Theory Volume

Conjugate Gravitational Coupling G_d

Johnson CQG **31** (2014) 205002

Dolan JHEP **1014** (2014) 179

Kastor/Ray/Traschen JHEP **1114** (2014) 120

Karch/Robinson JHEP **1015** (2015) 73

Holographic Smarr Relation

Karch/Robinson
JHEP **1512** (2015) 073

$$\ell^4 = \frac{\sqrt{2}\,\ell_{Pl}^4}{\pi^2} N$$

large N

$$\Omega(N,\mu,T,l) = N^2\Omega_0(\mu,T,l)$$

Free energy

$$E = TS + \tilde{\phi}\tilde{Q} + \Omega J + \tilde{\mu}C$$

$$E = (D-2)pV$$

CFT Eqn of State

polynomial functions

Sinamuli/Mann
PRD **96** (2017) 086008

$$\Omega(N,\mu,T,\alpha_j,R) = \sum_{k=0} g_k(N)\Omega^k(\mu,T,\alpha_j,R)$$

Extension to Lovelock Gravity

scaling: $\alpha_k = [L]^{D-2}$

$$g_k(N) = \beta_k(\alpha_k)^{\frac{D-2}{2(k-1)}}$$

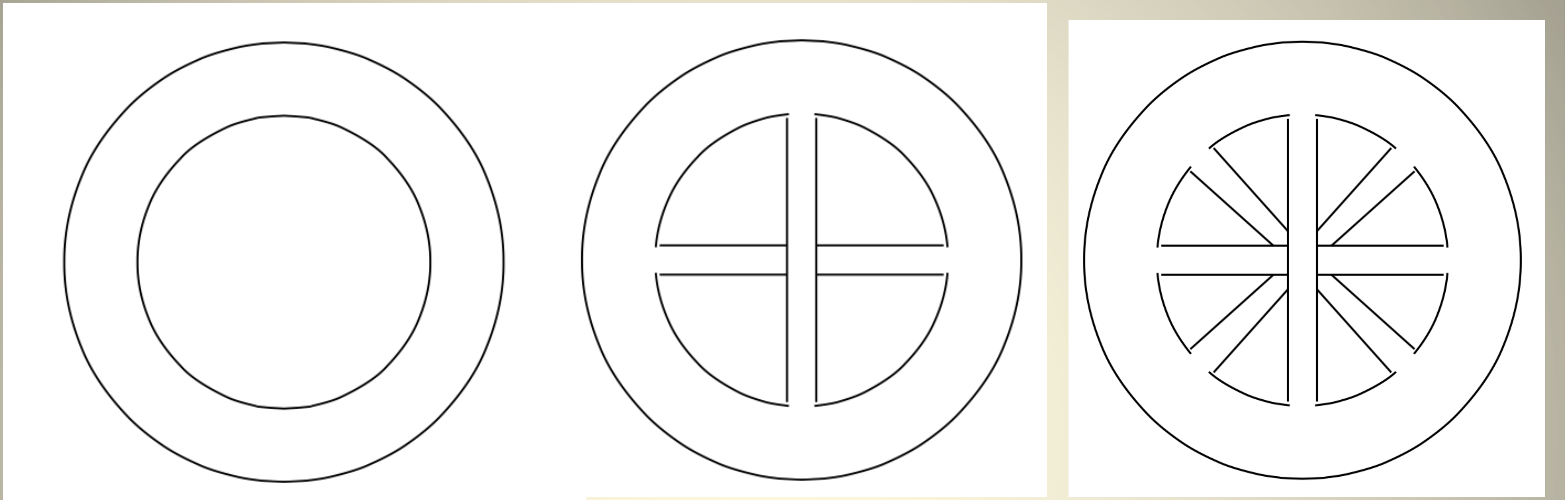
$$\ell\frac{\partial}{\partial\ell} + \sum_{k=1} 2(k-1)\alpha_k\frac{\partial}{\partial\alpha_k} = (d-2)\sum_{k=0} g_k\frac{\partial}{\partial g_k}$$

$$\Omega = \sum_{k=0} g_k\frac{\partial\Omega}{\partial g_k}$$

$$\frac{D-3}{D-2}M = TS - \frac{2}{D-2}PV + \sum_{k=2}^{K}\frac{2(k-1)}{D-2}\Psi^k\alpha_k + \frac{D-3}{D-2}\sum_j \Phi^j Q^j$$

Evaluating $g_k(N)$

Sinamuli/Mann
PRD **96** (2017) 086008



$N^2(g_{YM})^0 = N^2\lambda^0$
$\to \mathcal{L}^{(0)}$

$N^2(g_{YM})^4 = N^0\lambda^2$
$\to \mathcal{L}^{(1)}$

$N^2(g_{YM})^8 = N^{-2}\lambda^4$
$\to \mathcal{L}^{(2)}$

$$L^{(k)} \sim R^k \quad \to \quad N^{2(1-k)}$$

Holographic Complexity

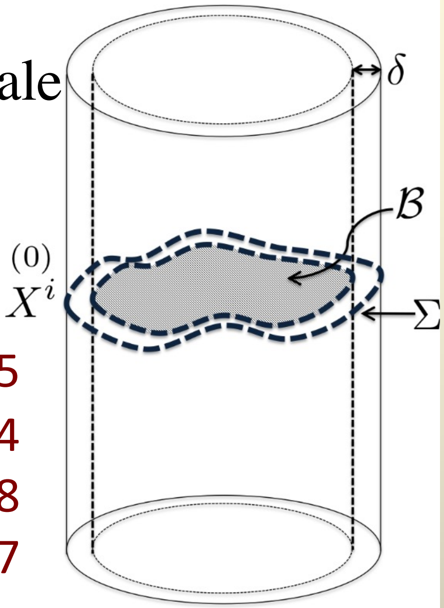
$\mathcal{C}(\Sigma)$: quantum complexity of boundry state at time slice Σ

Complexity = Volume

$$\mathcal{C}_{\mathcal{V}}(\Sigma) = \max_{\Sigma=\partial\mathcal{B}} \left[\frac{\mathcal{V}(\mathcal{B})}{G_N R} \right]$$

$\mathcal{V}(\mathcal{B})$: volume of extremal co-dim-1 slice $\mathcal{B}$

R: arbitrary length scale (eg AdS length)



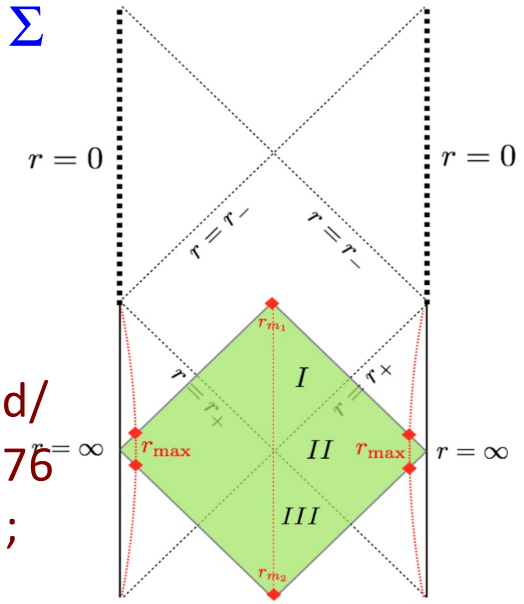
Susskind 1403.5695
Fortsch. Phys. 64 (2016) 24
Stanford/Susskind 1406.2678
PRD 90 (2014) 126007

Complexity= Action

$$\mathcal{C}_{\mathcal{A}}(\Sigma) = \frac{I_{\rm WDW}}{\pi}$$

WDW : Wheeler-de Witt patch domain of dependence of Cauchy slice Σ

I_{WDW} : action on this domain



Brown/Roberts/Susskind/
Swingle/Zhao 1509.07876
PRL 116 (2016) 191301 ;
PRD 93 (2016) 086006

Complexity and Black Hole Thermodynamics

Complexity of Formation

Lehner/Myers/Poisson/Sorkin
PRD **94** (2016) 084046

Complexity = Volume

$$\Delta\mathcal{C}_{\mathcal{V}}(\Sigma) = \lim_{r_{\max}\to\infty} \frac{\left[\mathcal{V}(\Sigma) - 2\mathcal{V}_{AdS}\right]}{G_N R}$$

Complexity= Action

$$\Delta\mathcal{C}_{\mathcal{A}}(\Sigma) = \frac{I_{\mathrm{WDW}} - 2I_{\mathrm{AdS}}}{\pi}$$

General Expectation (near extremality)

$$\Delta\mathcal{C}_{\mathcal{V}} \sim \tilde{k}_d S \log\frac{\mu}{T}$$

Carmi/Chapman/Marrochio
Myers/Sugishita
JHEP **2017** (2017)

$$\Delta\mathcal{C}_{\mathcal{A}} \sim k_d S \log\frac{\mu}{T}$$

Chapman/Heller/Marrochio/Pastawski
PRL **120** (2017) 121602

Based on calculations for charged spherically symmetric blackholes

Complexity and Black Hole Thermodynamics

Complexity of Formation

Complexity = Volume

$$\Delta\mathcal{C}_{\mathcal{V}}(\Sigma) = \lim_{r_{\max}\to\infty} \frac{\left[\mathcal{V}(\Sigma) - 2\mathcal{V}_{AdS}\right]}{G_N R}$$

Complexity= Action

$$\Delta\mathcal{C}_{\mathcal{A}}(\Sigma) = \frac{I_{\mathrm{WDW}} - 2I_{\mathrm{AdS}}}{\pi}$$

Balushi /Hennigar/Khunduri
PRL **126** (2021) 101601
JHEP **2105** (2021) 226

We find

$$\cancel{\Delta\mathcal{C}_{\mathcal{V}} \sim S\log\frac{\mu}{T}} \qquad \cancel{\Delta\mathcal{C}_{\mathcal{A}} \sim S\log\frac{\mu}{T}}$$

$$\Delta\mathcal{C}_{\mathcal{V}} \sim V^{\frac{D-2}{D-1}}\log\frac{\Omega_H}{T} \qquad \Delta\mathcal{C}_{\mathcal{A}} \sim V^{\frac{D-2}{D-1}}\log\frac{\Omega_H}{T}$$

where V is the thermodynamic volume of a rotating black hole

Complexity ~ Thermodynamic Volume

3+1 Dimensional Charged Black Holes

Entropy $S = \pi r_+^2$ Volume $V = \frac{4\pi r_+^3}{3}$

(Most) D-Dimensional Spherical Black Holes

Entropy $S \sim r_+^{D-2}$ Volume $V \sim r_+^{D-1}$

- Degeneracy between entropy and volume for most spherically symmetric black holes
- This degeneracy is broken for rotating black holes

Entropy $S \sim \pi\left(r_+^2 + a^2\right)$ Volume $V = \frac{4\pi l^2 r_+}{3} \frac{a^2 + r_+^2}{l^2 - a^2}$

$+ 4\pi a J / 3$

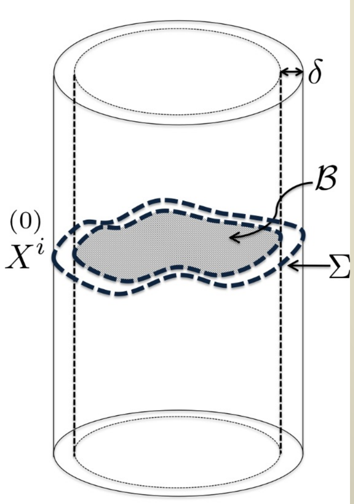
Cvetic/Gibbons/Kubiznak/Pope
PRD**84** (2011) 024037

Computing Complexity in Rotating Black Holes?

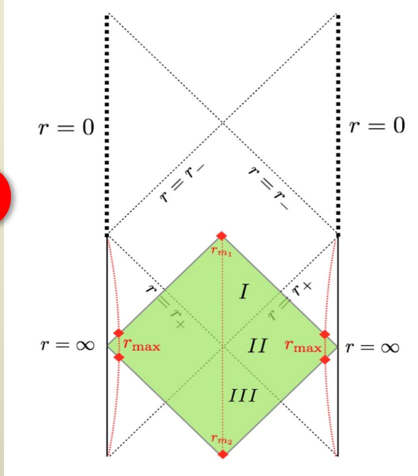
- Null hypersurfaces are complicated functions of polar angle as well as radial variable

AlBalushi/Mann
CQG **36** (2019) 245017

- Not clear how to match null hypersurfaces from AdS boundary to interior



?



Problem drastically simplifies for multiply-rotating black holes in odd-dimensions with equal angular momenta

- Caustics can form if charge is sufficiently large

Imseis/AlBalushi/Mann
CQG **38** (2021) 045018

Multiply Rotating AdS Black Holes in D=2N+3 Dimensions

$$ds^2 = -f(r)^2 dt^2 + g(r)^2 dr^2 + r^2 \hat{g}_{ab} dx^a dx^b$$

$$+ h(r)^2 \left[d\psi + A\left(x\right) - \Omega(r) dt \right]^2$$

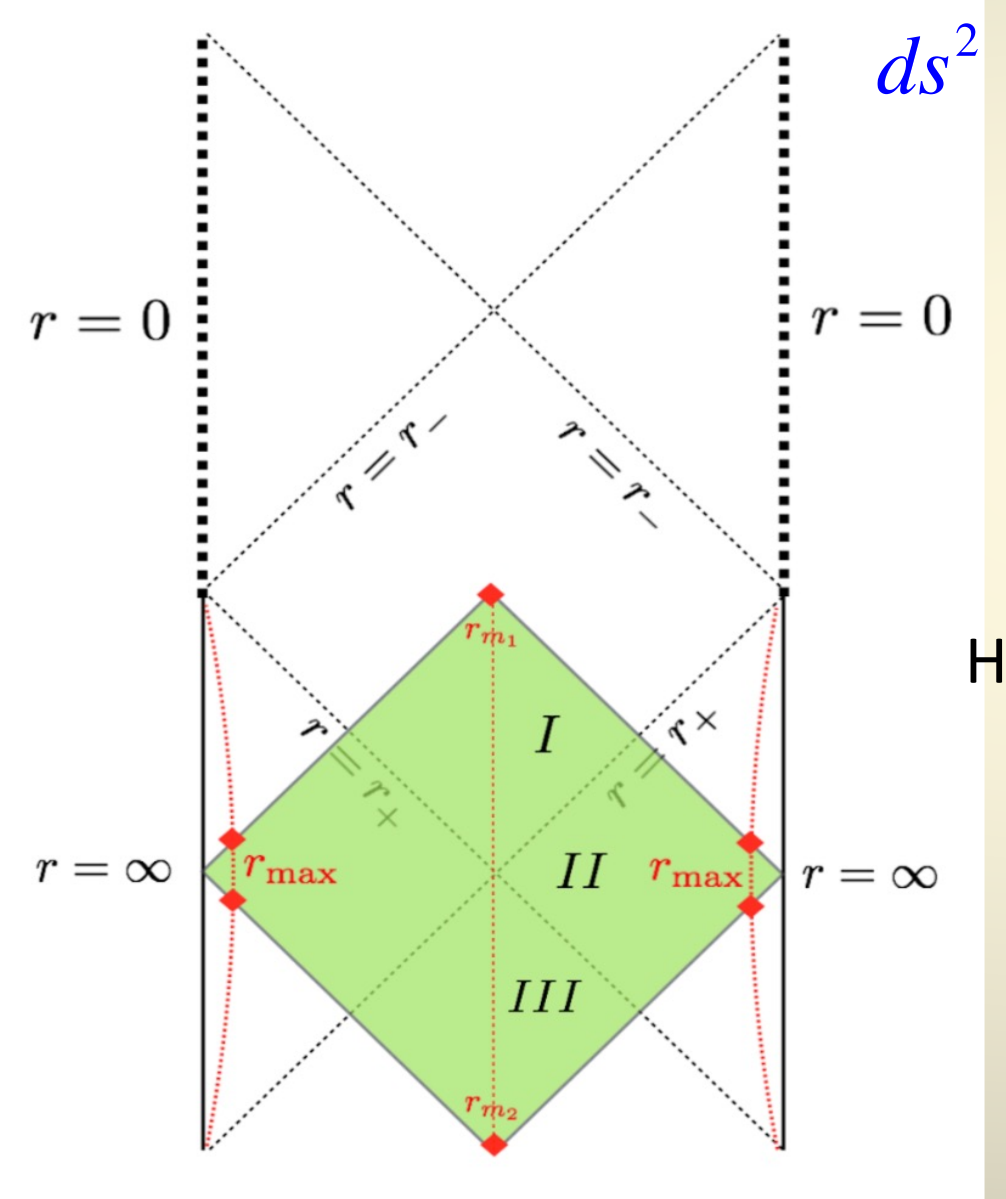
$$g(r)^2 = \left(1 + \frac{r^2}{\ell^2} - \frac{2m\Xi}{r^{2N}} + \frac{2ma^2}{r^{2N+2}} \right)^{-1} \qquad f(r) = \frac{r}{g(r)h(r)}$$

$$h(r)^2 = r^2 \left(1 + \frac{2ma^2}{r^{2N+2}} \right) \qquad \Omega(r) = \frac{2ma}{r^{2N}h^2} \qquad \Xi = 1 - \frac{a^2}{\ell^2}$$

$$\hat{R}_{ij} = 2(N+1)\hat{g}_{ij} \left\{ \begin{array}{l} D = 5 \\ \hat{g} = \frac{1}{4}\left(d\theta^2 + \sin^2\theta d\phi^2 \right) \\ A = \frac{1}{2}\cos\theta d\phi \end{array} \right.$$

Gibbons/Lu/Page/Pope [hep-th/0404008]
J Geom Phys (200549
Kunduri/Lucietti /Reall
[hep-th/0606076]PRD74 (2006) 084021

Causal Structure



$$ds^2 = -f(r)^2 dt^2 + g(r)^2 dr^2$$
$$+ r^2 \hat{g}_{ab} dx^a dx^b$$
$$+ h(r)^2 \left[d\psi + A - \Omega(r) dt \right]^2$$

Simple null hypersurfaces!

Horizon Generator

$$\xi = \frac{\partial}{\partial t} + \Omega_H \frac{\partial}{\partial \psi}$$

$$\Omega_H = \frac{2ma}{r_+^{2N+2} + 2ma^2}$$

Complexity equals Action

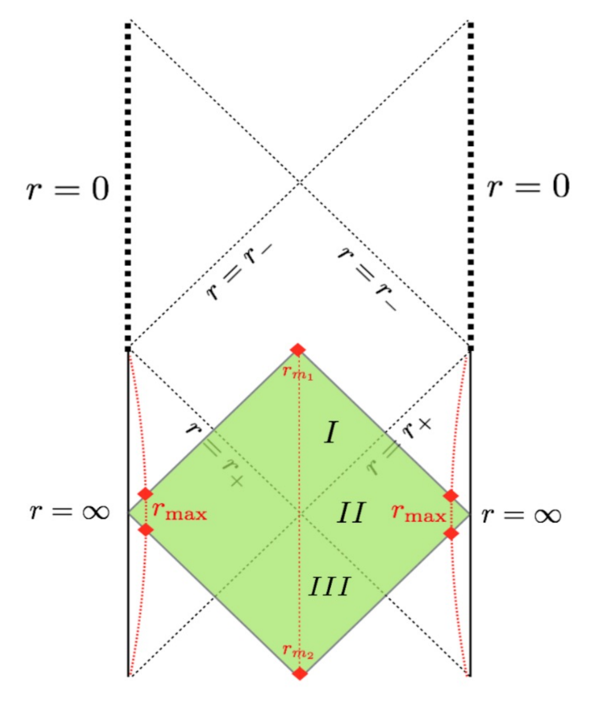
$$\mathcal{C}_{\mathcal{A}}(\Sigma) = \frac{I_{\text{WDW}}}{\pi}$$

$$\pi\Delta\mathcal{C}_{\mathcal{A}}(\mathrm{Y}) = I_{\text{WDW}} - 2I_{\text{AdS}}$$

$$= \frac{\Lambda\Omega_{2N+1}}{2(N+1)(2N+1)\pi G_N}\left[\int_{r_{m0}}^{\infty} dr\ r^{2N+1} g(r)^2 h(r) - \int_0^{\infty} dr \frac{r^{2(N+1)}}{1+r^2/\ell^2}\right]$$

$$-\frac{\Omega_{2N+1}(r_{m_0})^{2N+1}}{2\pi G_N(2N+1)} - \frac{\Omega_{2N+1}}{2\pi G_N}\int_{r_{m0}}^{\infty} dr\, r^{2N}\left[h(r)\frac{\Theta'}{\Theta}+1\right]$$

$$-\frac{\Omega_{2N+1}}{4\pi G_N}(r_{m_0})^{2N} h(r_{m_0})\log \ell_{\text{ct}}^2\Theta(r_{m_0})^2 \,|\, f(r_{m_0})^2 \,|$$



$$\Theta = \frac{1}{f(r)g(r)}\left[\frac{2N}{r} + \frac{h'}{h}\right]$$

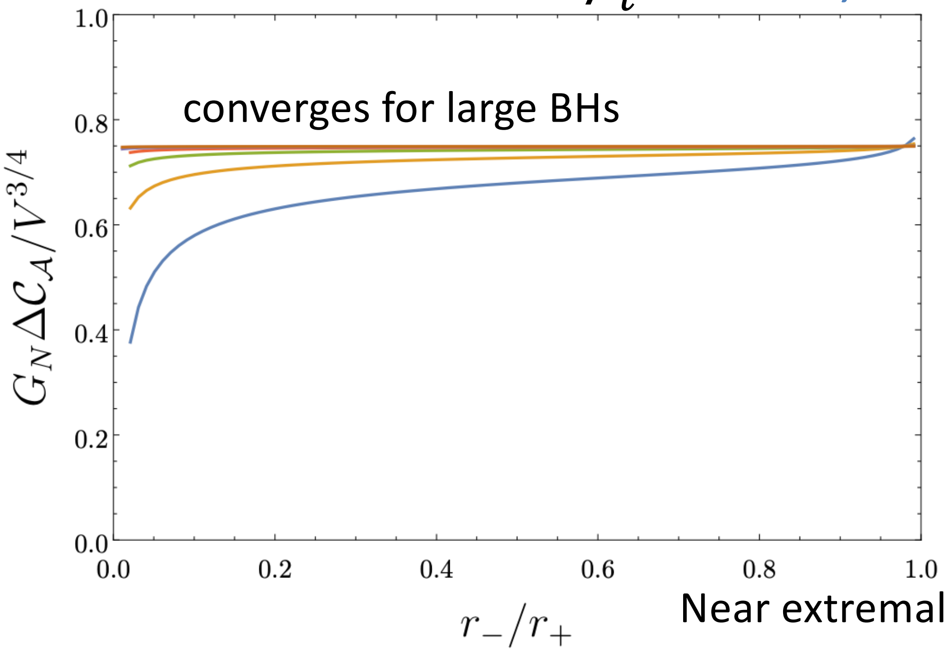
Balushi /Hennigar/Khunduri
PRL **126** (2021) 101601
JHEP **2105** (2021) 226

We find

$$\Delta\mathcal{C}_{\mathcal{A}}(\Sigma) = \Sigma_{\text{g}} C_T \left(\frac{V}{V_{\text{AdS}}}\right)^{\frac{D-2}{D-1}}$$

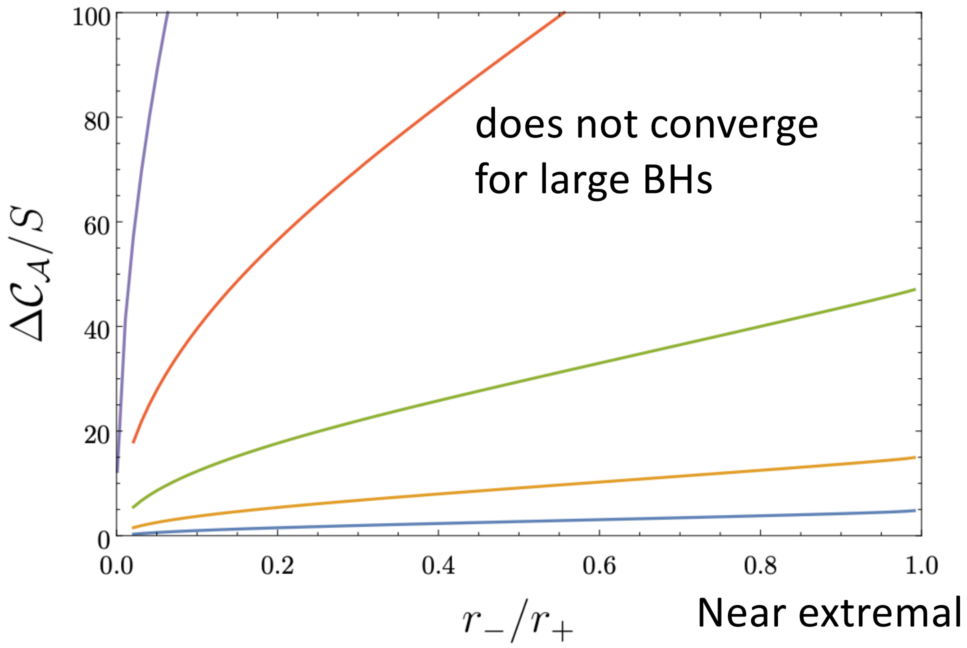
D=5

$r_+/\ell = 10^3, 10^4, 10^5, 10^6, 10^7$



converges for large BHs

Near extremal



does not converge for large BHs

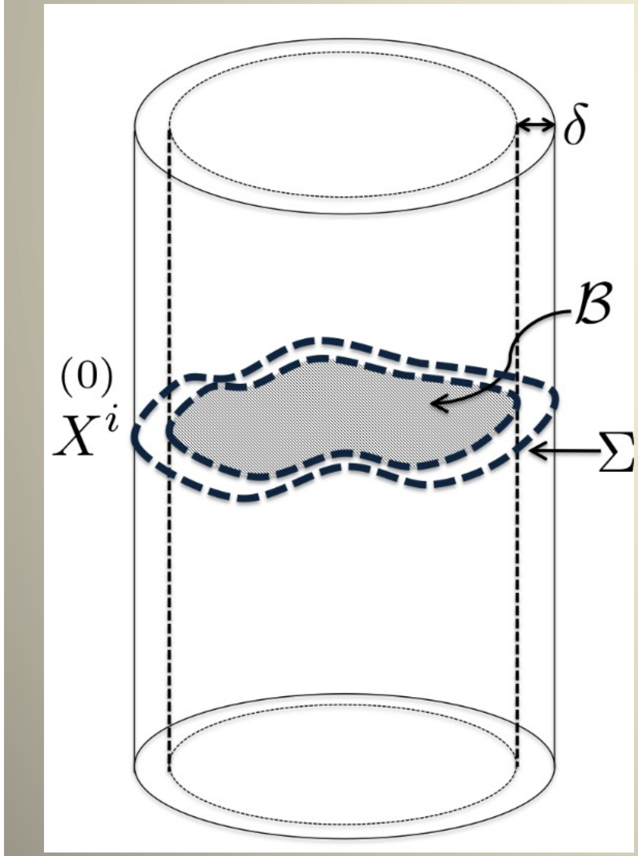
Near extremal

$$\Delta\mathcal{C}_{\mathcal{V}} = \Sigma_g C_T \left(\frac{V}{V_{\mathrm{AdS}}}\right)^{\frac{D-2}{D-1}} = \Sigma_g C_T \left(\frac{V}{V_{\mathrm{AdS}}}\right)^{\frac{3}{4}}$$

$$\neq KS$$

Complexity equals Volume

$$\mathcal{C}_{\mathcal{V}}(\Sigma) = \max_{\Sigma=\partial\mathcal{B}}\left[\frac{\mathcal{V}(\mathcal{B})}{G_N R}\right]$$



$$\mathcal{V} = [\mathcal{V} - 2\mathcal{V}_{AdS}]$$

$$= 2\Omega_{D-2}\int_{r_+}^{r_{\max}} dr\, r^{(D-3)} h(r) g(r)$$

$$- 2\Omega_{D-2}\int_0^{r_{\max}} \frac{r^{D-2}dr}{\sqrt{1+r^2/\ell^2}}$$

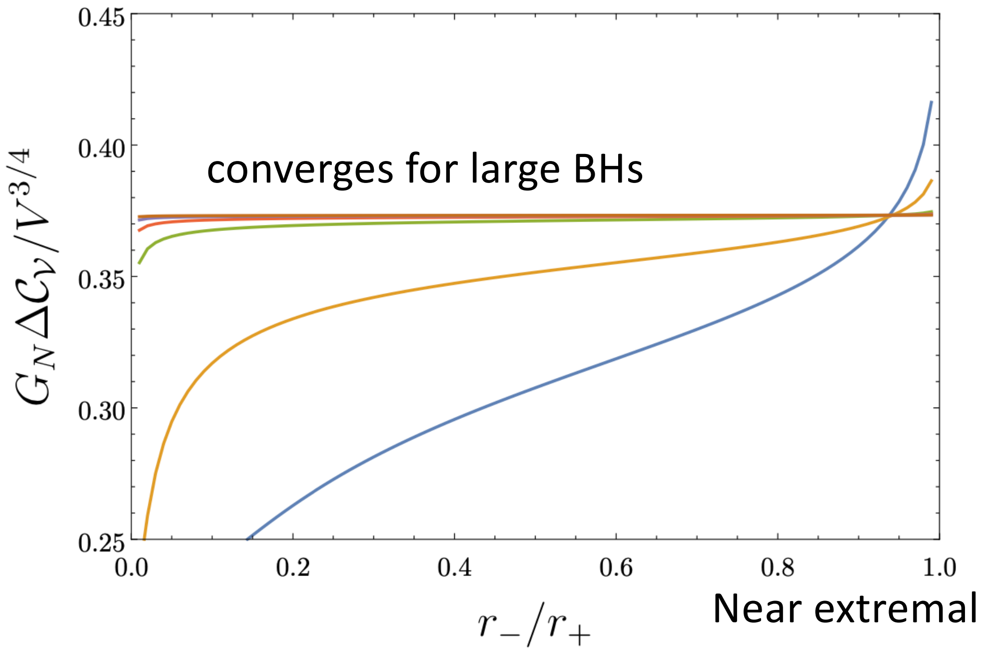
Balushi /Hennigar/Khunduri
PRL **126** (2021) 101601
JHEP **2105** (2021) 226

We find

$$\Delta\mathcal{C}_{\mathcal{V}} = \lim_{r_{\max}\to\infty}\frac{[\mathcal{V}-2\mathcal{V}_{AdS}]}{G_N R} = \Sigma_g C_T \left(\frac{V}{V_{\mathrm{AdS}}}\right)^{\frac{D-2}{D-1}}$$

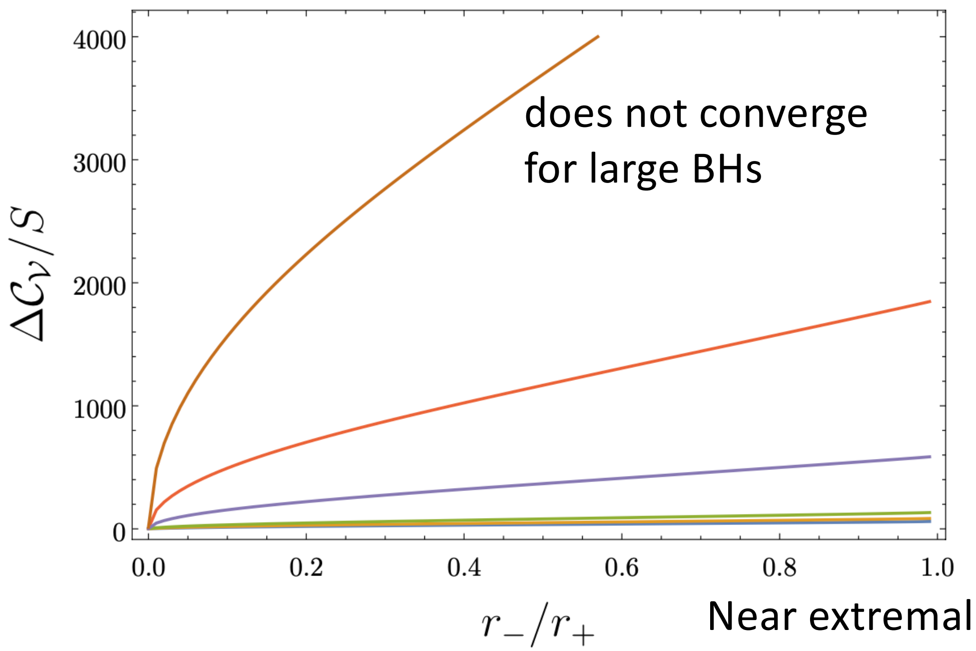
D=5

$r_+/\ell = 10, 10^2, 10^3, 10^4, 10^5, 10^6$



converges for large BHs

Near extremal



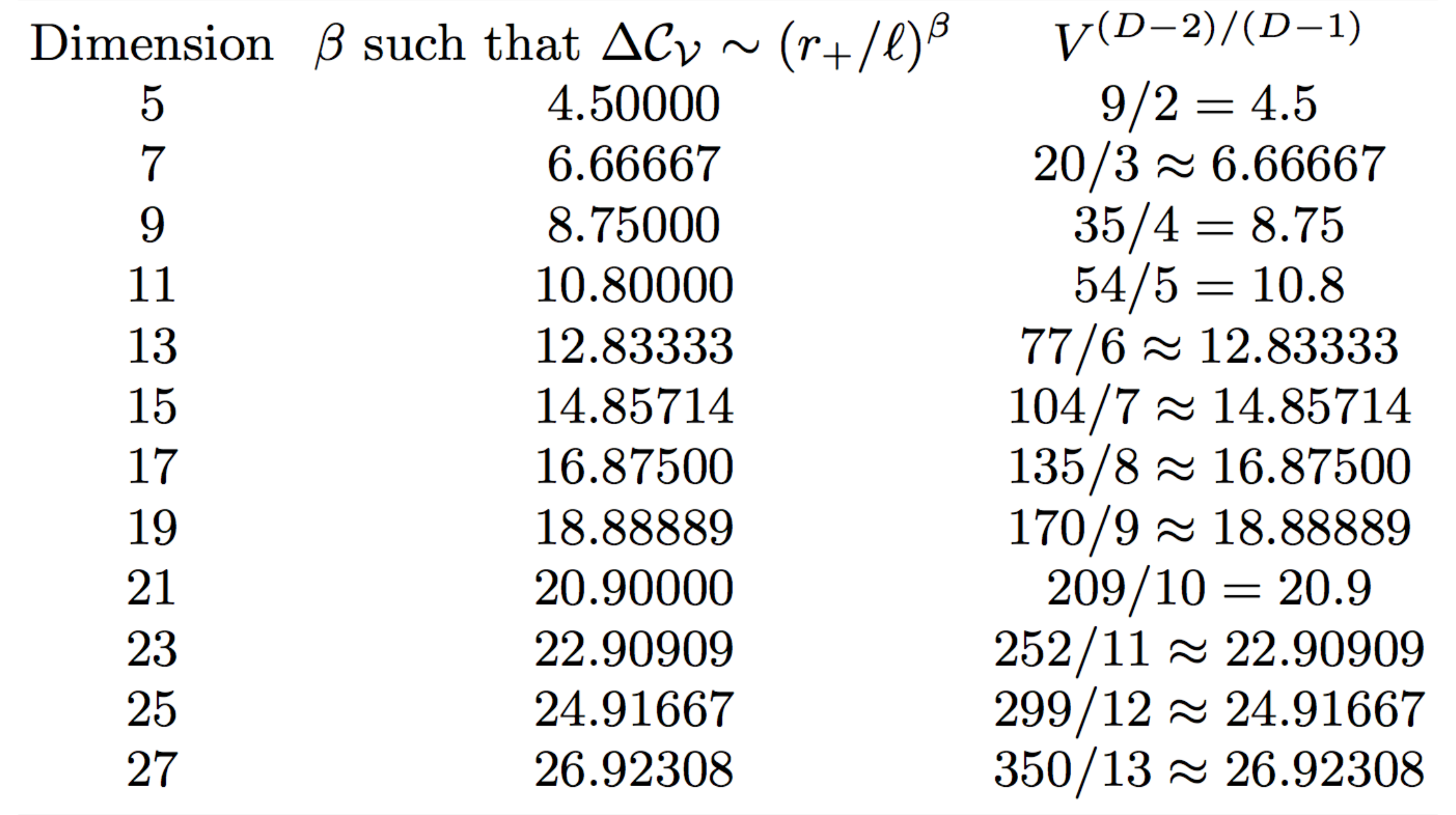
does not converge for large BHs

Near extremal

$$\Delta\mathcal{C}_\mathcal{V} = \Sigma_g C_T \left(\frac{V}{V_{\mathrm{AdS}}}\right)^{\frac{D-2}{D-1}} = \Sigma_g C_T \left(\frac{V}{V_{\mathrm{AdS}}}\right)^{\frac{3}{4}}$$

$$\neq KS$$

Holds for all higher odd dimensions



Dimension	β such that $\Delta\mathcal{C}_{\mathcal{V}} \sim (r_+/\ell)^\beta$	$\mathcal{V}^{(D-2)/(D-1)}$
5	4.50000	$9/2 = 4.5$
7	6.66667	$20/3 \approx 6.66667$
9	8.75000	$35/4 = 8.75$
11	10.80000	$54/5 = 10.8$
13	12.83333	$77/6 \approx 12.83333$
15	14.85714	$104/7 \approx 14.85714$
17	16.87500	$135/8 \approx 16.87500$
19	18.88889	$170/9 \approx 18.88889$
21	20.90000	$209/10 = 20.9$
23	22.90909	$252/11 \approx 22.90909$
25	24.91667	$299/12 \approx 24.91667$
27	26.92308	$350/13 \approx 26.92308$

Thermodynamic Volume

$$V = \frac{r_+^{2(N+1)}\Omega_{2N+1}}{2(N+1)} + \frac{4\pi a J}{(2N+1)(N+1)}$$

Thermodynamic Entropy

$$S = \frac{\Omega_{2N+1} h(r_+) r_+^{2N}}{4G_N}$$

Lesson: Complexity of Formation scales with Thermodynamic Volume

Complexity = Volume

Complexity= Action

$$\Delta\mathcal{C}_{\mathcal{V}}(\Sigma) = \lim_{r_{\max}\to\infty} \frac{\left[\mathcal{V}(\Sigma) - 2\mathcal{V}_{AdS}\right]}{G_N R}$$

$$\Delta\mathcal{C}_{\mathcal{A}}(\Sigma) = \frac{I_{\text{WDW}} - 2I_{\text{AdS}}}{\pi}$$

$$\Delta\mathcal{C}_{\mathcal{V}} = \Sigma_{\text{g}} C_T \left(\frac{V}{V_{\text{AdS}}}\right)^{\frac{D-2}{D-1}}$$

$$\Delta\mathcal{C}_{\mathcal{A}}(\Sigma) = \Sigma_{\text{g}} C_T \left(\frac{V}{V_{\text{AdS}}}\right)^{\frac{D-2}{D-1}}$$

Thermodynamic Volume

$$V = \frac{r_+^{2(N+1)}\Omega_{2N+1}}{2(N+1)} + \frac{4\pi a J}{(2N+1)(N+1)}$$

Also for Kerr-AdS (partial results) and BTZ!

Benamonti/Bigazzi/BilloFaggi/Galli
JHEP **2111** (2021) 037

Thermodynamic Entropy

$$S = \frac{\Omega_{2N+1} h(r_+) r_+^{2N}}{4G_N}$$

And solitons!

Andrews/Hennigar/Kunduri
CQG **37** (2020) 204002

Entropy Bounds Complexity

Reverse Isoperimetric Inequality

$$\mathcal{R}^{D-2} = \left(\frac{(D-1)V}{\Omega_{D-2}}\right)^{(D-2)/(D-1)} \left(\frac{\Omega_{D-2}}{4G_N S}\right) \geq 1$$

$$\Delta\mathcal{C} \geq \beta_D S$$

Complexity scales with volume
but is bounded from below by black hole entropy

Central Charge Criticality

Cong/Kubiznak/Mann
PRL **127** (2021) 091301

Recall Holographic Smarr Relation

$$E = TS + \tilde{\phi}\tilde{Q} + \Omega J + \tilde{\mu} C$$

Dimensional Scaling suggests

$$C = k\frac{l^{D-2}}{16\pi G}$$

CFT 1st Law

$$\delta E = T\delta S - p d\mathcal{V} + \tilde{\phi}\delta\tilde{Q} + \Omega\delta J + \tilde{\mu}\delta C$$

$$E = M$$

$$\tilde{Q} = \frac{Ql}{\sqrt{G}}$$

$$\tilde{\phi} = \frac{\phi\sqrt{G}}{l}$$

$$\delta M = \frac{\kappa}{8\pi G}\delta A + \Omega\delta J + \phi\delta Q - \frac{V}{8\pi G}\delta\Lambda - \alpha\frac{\delta G}{G}$$

Bulk 1st Law

$$\begin{aligned} V &= \frac{8\pi G l^2}{(D-1)(D-2)}\Big(M - \phi Q - (D-2)C\tilde{\mu}\Big) \\ &= \frac{(D-3)}{2P}\left(\frac{D-2}{D-3}(\Omega J + TS) + \phi Q - M\right) \end{aligned}$$

Bulk Smarr Relation

Bulk 1st Law

Variable G

$$\delta M = \frac{\kappa}{8\pi G}\delta A + \Omega\delta J + \phi\delta Q - \frac{V}{8\pi G}\delta\Lambda - \alpha\frac{\delta G}{G}$$

$$\begin{aligned}\alpha &= \frac{1}{2}\phi Q + \tilde{\mu}C + TS \\ &= M - \Omega J - \frac{1}{2}\phi Q\end{aligned}$$

Conjugate to G

Rewrite in terms of Pressure

$$\frac{\delta G}{G} = -\frac{2}{D}\frac{\delta C}{C} - \frac{D-2}{D}\frac{\delta P}{P}$$

$$\delta M = T\delta S + \Omega\delta J + \phi\delta Q + V_C\delta P + \mu\delta C$$

“Mixed” Bulk 1st Law

New Thermodynamic Volume

$$V_C = \frac{2M + (D-4)\phi Q}{2DP}$$

$$\mu = \frac{2P(V_C - V)}{C(D-2)}$$

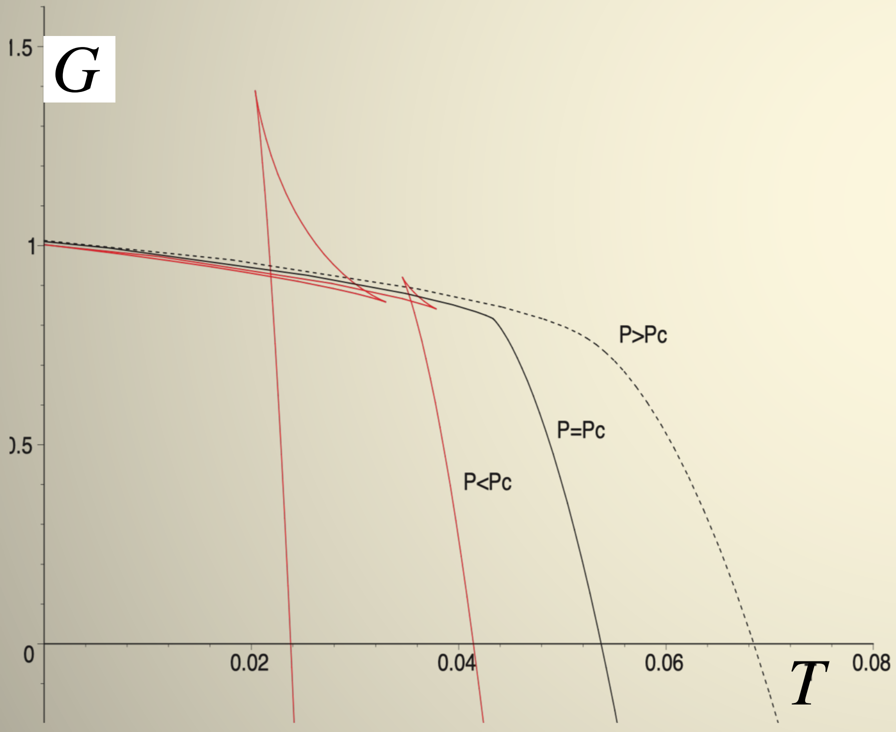
Conjugate Chemical Potential

Satisfies Reverse Isoperimetric Inequality for negative Chemical Potential

Charged AdS Black Holes

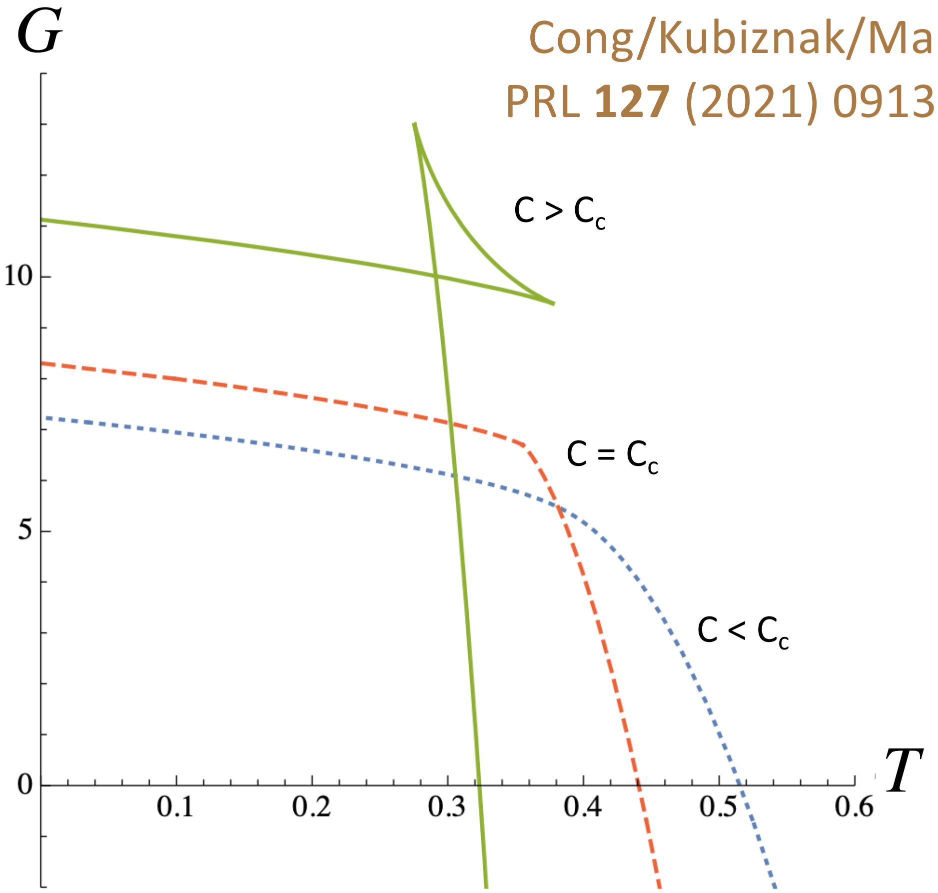
$$G = M - TS = \frac{3GQ^2l^2 + l^2r_+^2 - r_+^4}{4Gr_+l^2} = \frac{2\pi r_+}{3}\sqrt{\frac{6PC}{k}} - \frac{2\pi P r_+^3}{3} + \frac{3Q^2}{4r_+}$$

Kubiznak/Mann
JHEP **1207**(2012) 033



Bulk: Critical Pressure

Cong/Kubiznak/Mann
PRL **127** (2021) 091301



Boundary: Critical Central Charge

Visser PRD **105** (2022) 106014
Cong/Kubiznak/Mann/Visser
JHEP **08** (2022) 174

CFT Ensembles

No Interesting Phase Behaviour

fixed $(\tilde{Q}, p, C)$: $F_1 \equiv E - TS + p\mathcal{V} = \tilde{\Phi}\tilde{Q} + \mu C + p\mathcal{V}\,,$

fixed $(\tilde{Q}, p, \mu)$: $F_2 \equiv E - TS + p\mathcal{V} - \mu C = \tilde{\Phi}\tilde{Q} + p\mathcal{V}\,,$

fixed $(\tilde{\Phi}, p, \mu)$: $F_3 \equiv E - TS - \tilde{\Phi}\tilde{Q} + p\mathcal{V} - \mu C = p\mathcal{V}\,,$

fixed $(\tilde{\Phi}, p, C)$: $F_4 \equiv E - TS - \tilde{\Phi}\tilde{Q} + p\mathcal{V} = \mu C + p\mathcal{V}\,,$

fixed $(\tilde{\Phi}, \mathcal{V}, \mu)$: $F_5 \equiv E - TS - \tilde{\Phi}\tilde{Q} - \mu C = 0\,.$

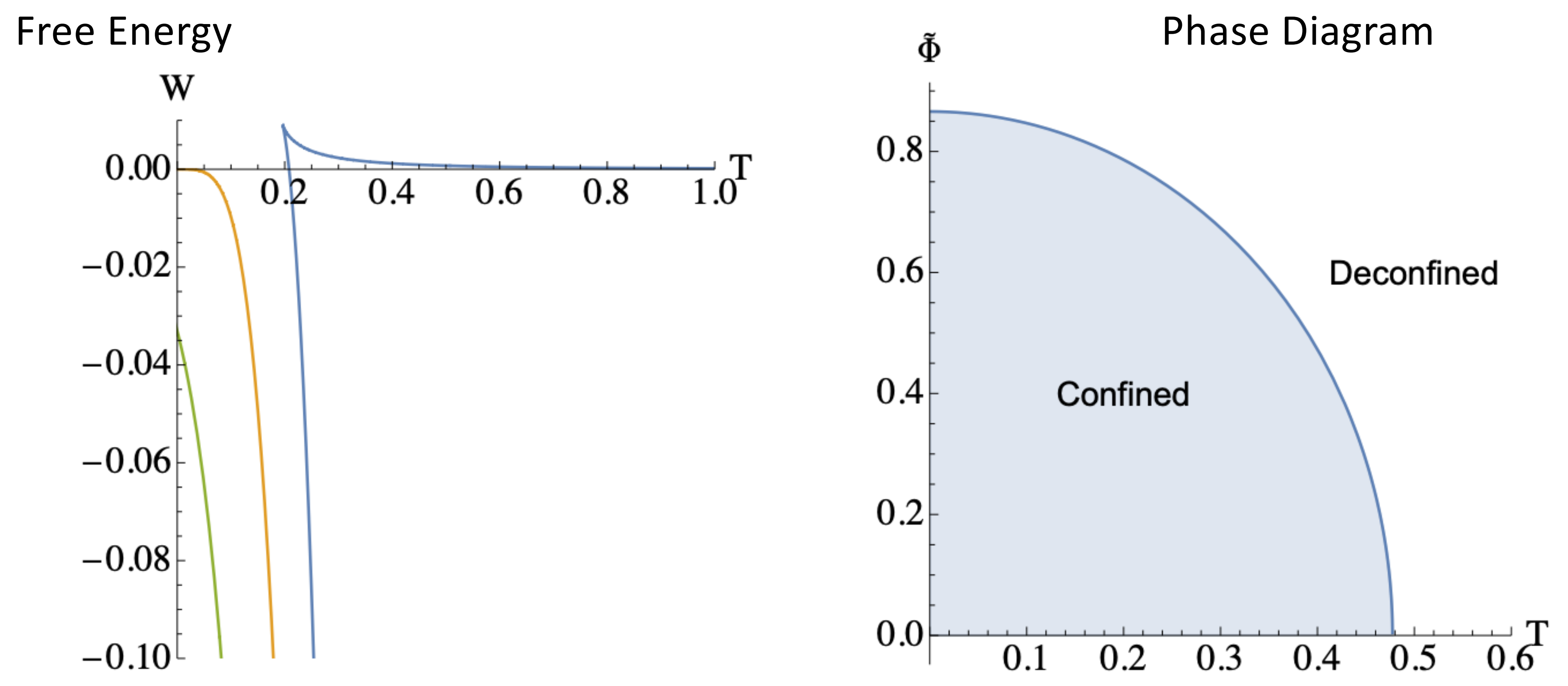
Interesting Phase Behaviour

fixed $(\tilde{Q}, \mathcal{V}, C)$: $F \equiv E - TS = \tilde{\Phi}\tilde{Q} + \mu C\,,$

fixed $(\tilde{\Phi}, \mathcal{V}, C)$: $W \equiv E - TS - \tilde{\Phi}\tilde{Q} = \mu C\,,$

fixed $(\tilde{Q}, \mathcal{V}, \mu)$: $G \equiv E - TS - \mu C = \tilde{\Phi}\tilde{Q}$

fixed $(\tilde{\Phi}, \mathcal{V}, C)$: $$W \equiv E - TS - \tilde{\Phi}\tilde{Q} = \mu C\,,$$



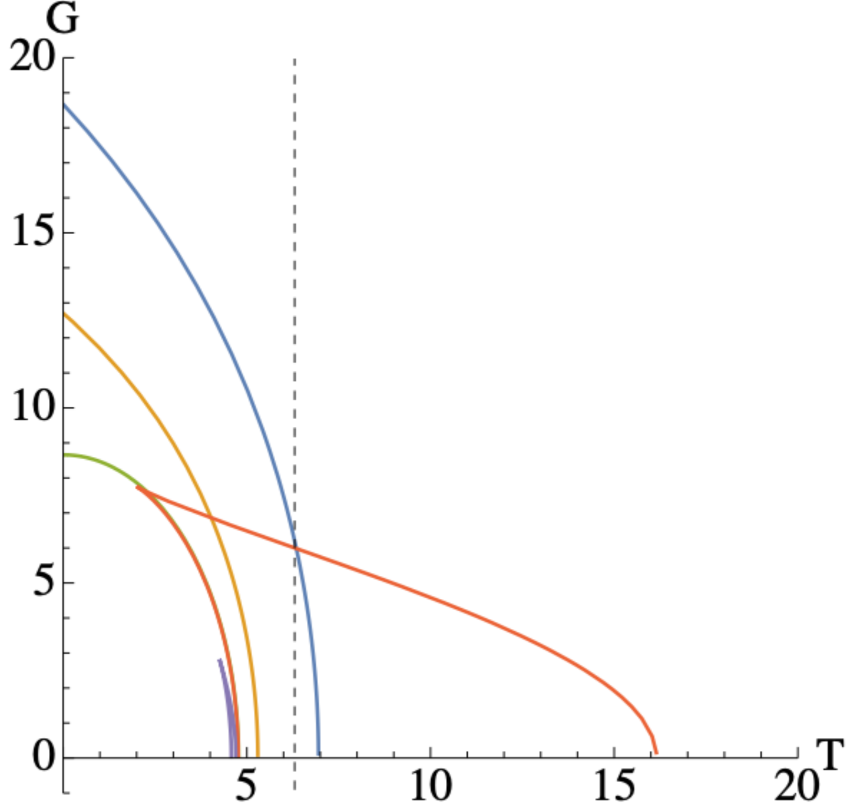
Free Energy

Phase Diagram

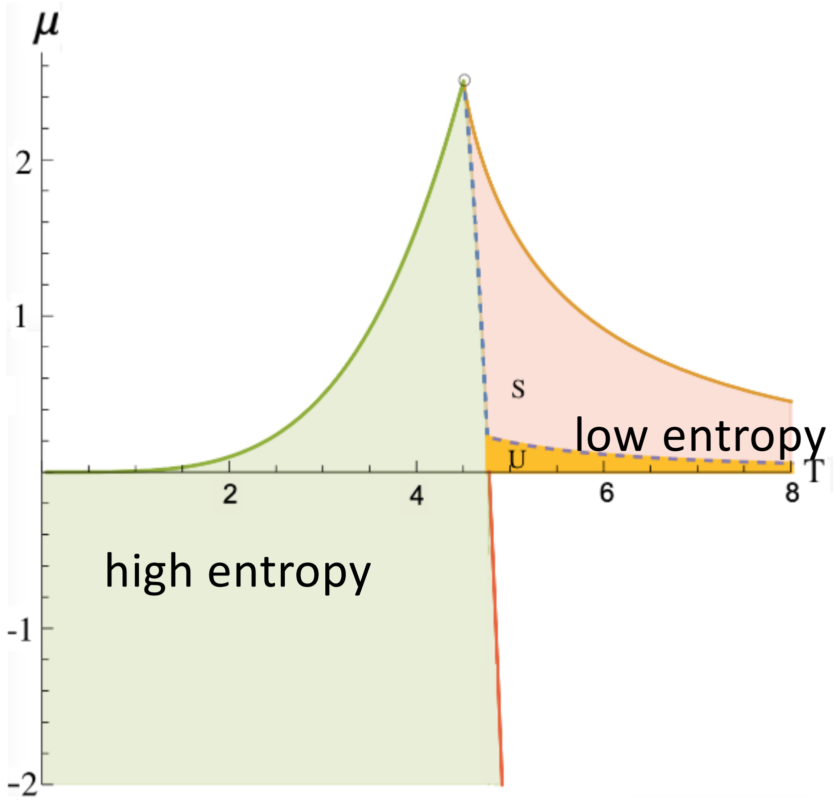
fixed $(\tilde{Q}, \mathcal{V}, \mu)$:

$$G \equiv E - TS - \mu C = \tilde{\Phi}\tilde{Q}$$

Free Energy



Phase Diagram



Holographic Dual of Black Hole Chemistry

Belhaj/Cong/Kubiznak/
Mann/Visser
PRL **130** (2023) 181401

Recall

$$C \propto \frac{L^{d-2}}{G_N}$$

Boundary metric

$$ds^2 = \omega^2\left(-dt^2 + L^2 d\Omega^2_{k,d-2}\right)$$

Standard choice:

CFT volume

$$\omega = 1 \Rightarrow \mathcal{V} \propto L^{d-2}$$

$\mu\delta C$ and $-p\delta\mathcal{V}$ not truly independent

or

variable G_N

Instead

ω as a thermodynamic variable

CFT volume $\mathcal{V} \propto (\omega L)^{d-2}$

Revised Holographic Dictionary

$$\tilde{S} = S = \frac{A}{4G_N} \quad \tilde{E} = \frac{M}{\omega} \quad \tilde{T} = \frac{T}{\omega} \quad \tilde{\Omega} = \frac{\Omega}{\omega}$$

$$\tilde{J} = J \quad \tilde{\Phi} = \frac{\Phi\sqrt{G_N}}{\omega L} \quad \tilde{Q} = \frac{QL}{\sqrt{G_N}}$$

Bulk 1st Law

$$\delta M = T\delta S + V\delta P + \Phi\delta Q + \Omega\delta J$$

Smarr also OK!

Full Duality

No Degeneracy

G fixed

Boundary 1st Law

$$\delta\tilde{E} = \tilde{T}\delta S + \tilde{\Omega}\delta J + \tilde{\Phi}\delta\tilde{Q} + \mu\delta C - p\delta\mathcal{V}$$

$$\mu = \frac{1}{C}(\tilde{E} - \tilde{T}S - \tilde{\Omega}J - \tilde{\Phi}\tilde{Q}) \qquad p = \frac{\tilde{E}}{(d-2)\mathcal{V}}$$

CFT Criticality: Open Questions

- Duals of Re-entrant Phase Transitions, Triple Points, Superfluids,….
- Higher Curvature terms?
- How to incorporate 1/N corrections?

Multicritical Behaviour

- Multicritical Behaviour
  - Multiple Phases become indistinguishable at a particular set of thermodynamic parameters (P,T)
- Do black holes exhibit such behaviour?
- Gibbs Phase Rule
  - Constrains the number of phases in terms of the number of thermodynamic parameters

$$F = W - P + 1$$

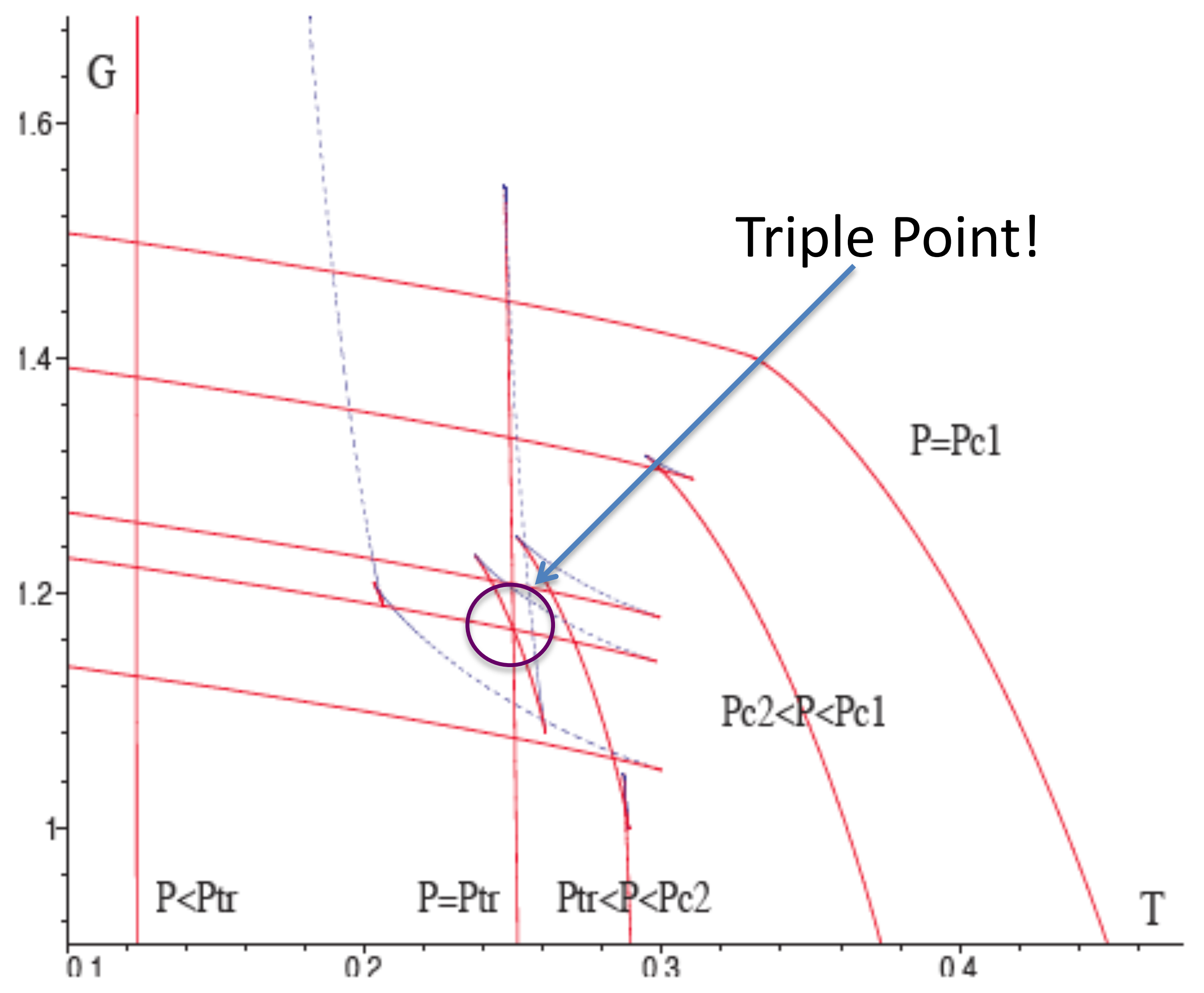
F: number of independent intensive parameters

W: number of conjugate pairs

P: number of coexistent phases

Black Hole Multicriticality

Recall: Doubly rotating black hole



Triple Point!

$$\frac{J_2}{J_1} = 0.05$$

$$P_{c1} = 0.259$$

$$P_{c1} = 0.0956$$

Altimirano/Kubiznak/
Sherkatgnad/RBM
CQG **31** (2014) 042001

Multicriticality in Non-linear Electrodynamics

Tavakoli/Wu/RBM
JHEP **2212** (2022) 117

$$S=\int d^4x\sqrt{-g}\left(R-2\Lambda-\sum_{i=1}^{N}\alpha_i(F^2)^i\right)$$

$$ds^2=-U(r)dt^2+\frac{1}{U(r)}dr^2+r^2d\Omega_2^2$$

$$A_\mu=[\Phi(r),0,0,0]$$

$$(r(U(r)-1))'+r^2\Lambda-r^2\sum_{n=1}^{N}\left(n-\frac{1}{2}\right)\alpha_n\left(-2(\Phi')^2\right)^n=0$$

$$\frac{1}{2}r^2\sum_{n=1}^{N}n\alpha_n\left(-2(\Phi')^2\right)^n-Q(\Phi')=0$$

$$\Phi=\sum_{i=1}^{K}b_ir^{-i}\qquad U=1+\sum_{i=1}^{K}c_ir^{-i}+\frac{r^2}{l^2}$$

$$c_1=-2M\qquad c_i=\frac{4Q}{i+2}b_{i-1}$$

$$b_1=Q\qquad b_5=\frac{4}{5}Q^3\alpha_2$$

$$b_9=\frac{4}{3}Q^5(4\alpha_2^2-\alpha_3)$$

$$b_{13}=\frac{32}{13}Q^7(24\alpha_2^3-12\alpha_3\alpha_2+\alpha_4)$$

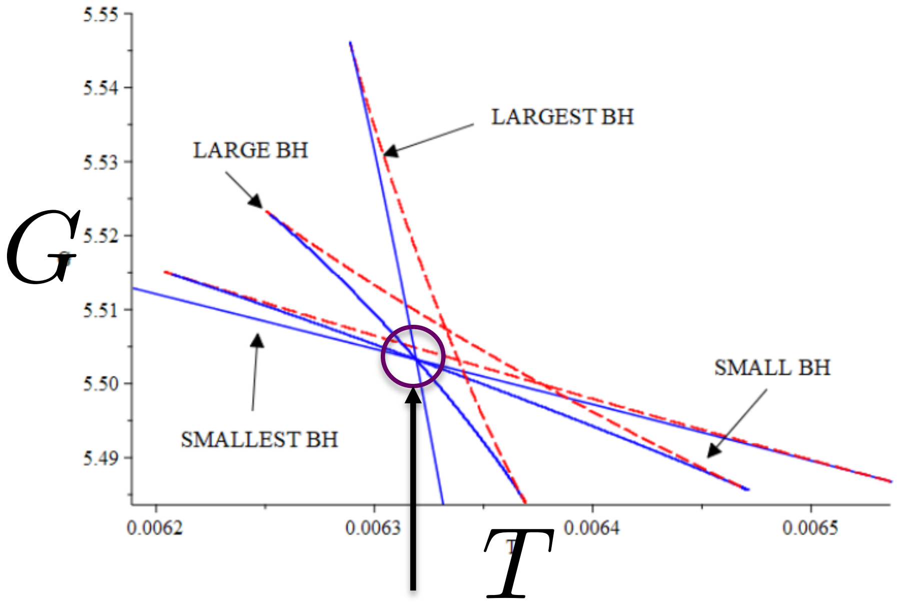
$$P = \frac{T}{2r_+} - \frac{1}{8\pi r_+^2} + \frac{Q^2}{8\pi r_+^4} + \frac{5b_5 Q}{16\pi r_+^8} + \frac{3b_9 Q}{8\pi r_+^{12}} + \frac{13b_{13} Q}{32\pi r_+^{16}} + \frac{17b_{17} Q}{40\pi r_+^{20}} + \frac{7b_{21} Q}{16\pi r_+^{24}} + \frac{25b_{25} Q}{56\pi r_+^{28}}$$

$P = 6.9 \times 10^{-5}\, Q = 6.75112$

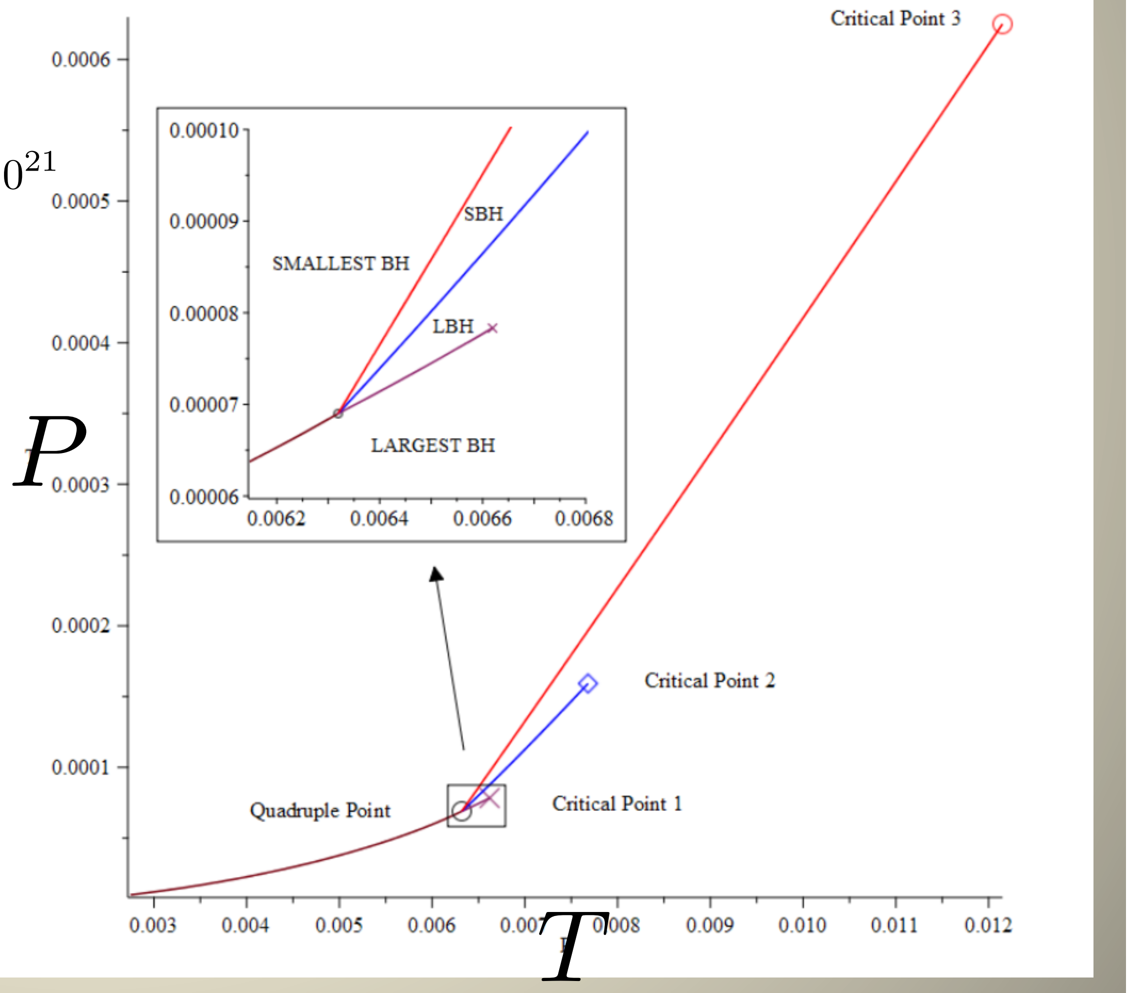
$\alpha_2 = -20.63286635\, \alpha_3 = 1379.056050$

$\alpha_4 = -133263.0329\, \alpha_5 = 1.550137197 \times 10^7$

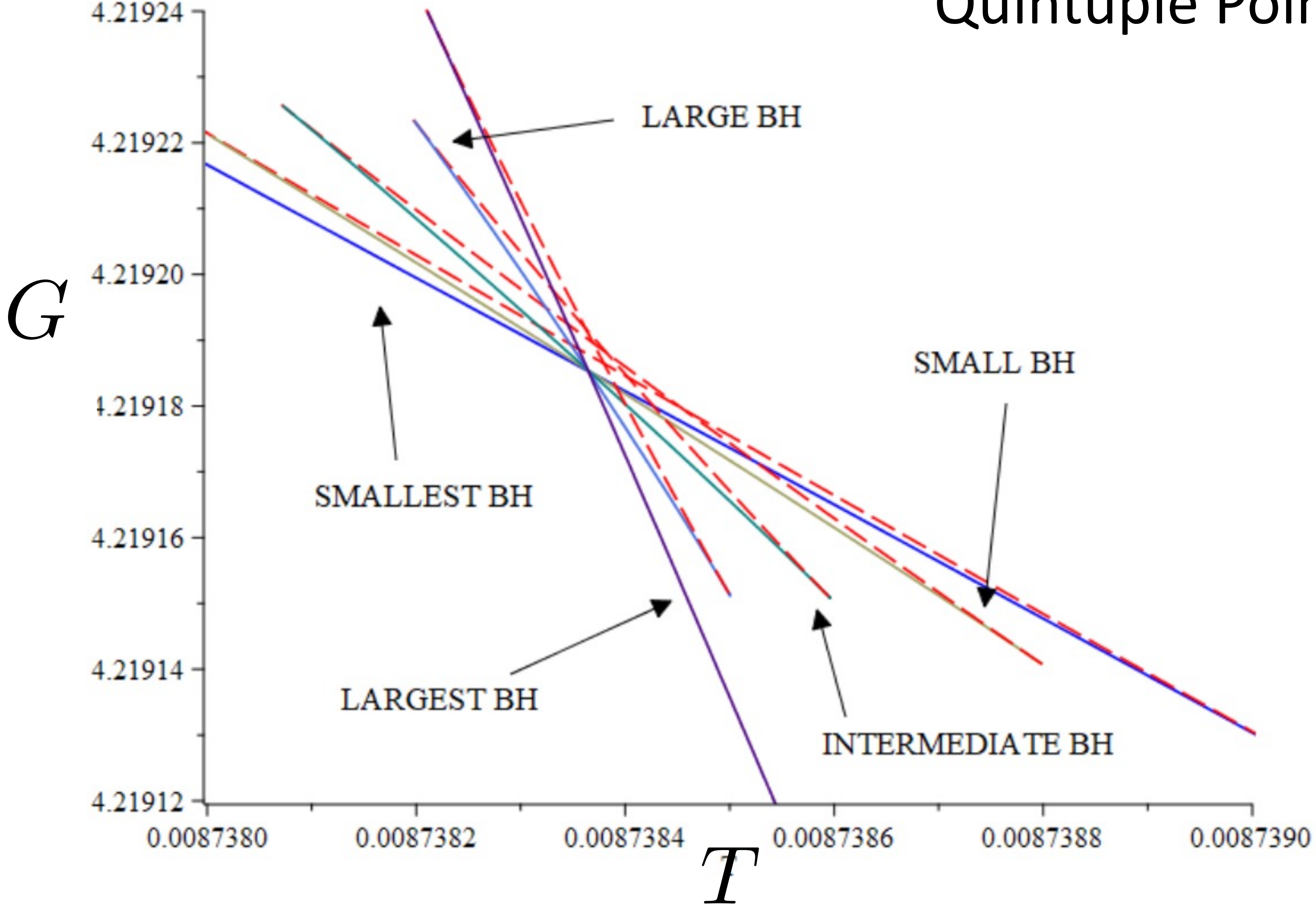
$\alpha_6 = -2.017480713 \times 10^9\, \alpha_7 = 5.046133724 \times 10^{21}$



Quadruple Point



Quintuple Point



Multicriticality in Multiply Rotating Black Holes

$$ds^2 = -W\left(1+\frac{r^2}{l^2}\right)d\tau^2 + \frac{2m}{U}\left(W d\tau - \sum_{i=1}^{N}\frac{a_i\mu_i^2 d\varphi_i}{\Xi_i}\right)^2$$

Kerr-AdS

$$+\sum_{i=1}^{N}\frac{r^2+a_i^2}{\Xi_i}\mu_i^2 d\varphi_i^2 + \frac{U dr^2}{F-2m} + \sum_{i=1}^{N+\epsilon}\frac{r^2+a_i^2}{\Xi_i}d\mu_i^2$$

$$-\frac{l^{-2}}{W(1+r^2/l^2)}\left(\sum_{i=1}^{N+\epsilon}\frac{r^2+a_i^2}{\Xi_i}\mu_i d\mu_i\right)^2$$

$$n = \frac{1}{2}(d-1-\epsilon)$$

$$W = \sum_{i=1}^{N+\epsilon}\frac{\mu_i^2}{\Xi_i} \qquad U = r^{\epsilon}\sum_{i=1}^{N+\epsilon}\frac{\mu_i^2}{r^2+a_i^2}\prod_{j}^{N}(r^2+a_j^2)$$

$$\epsilon = 0/1\,\mathrm{odd/even}\,D$$

$$\sum_{i=1}^{n}\mu_i^2 = 1$$

$$F = r^{\epsilon-2}\left(1+\frac{r^2}{l^2}\right)\prod_{i=1}^{N}(r^2+a_i^2) \qquad \Xi_i = 1-\frac{a_i^2}{l^2}$$

Wu/RBM
CQG Lett **40** (2023) 06LT01

$$M = \frac{m\omega_{D-2}}{4\pi(\prod_j \Xi_j)}(\sum_{i=1}^{N} \frac{1}{\Xi_i} - \frac{1-\epsilon}{2})$$

$$N = \frac{1}{2}(D - 1 - \epsilon)$$

$$J_i = \frac{a_i m\omega_{D-2}}{4\pi\Xi_i(\prod_j \Xi_j)} \qquad \Omega_i = \frac{a_i(1 + \frac{r_+^2}{l^2})}{r_+^2 + a_i^2}$$

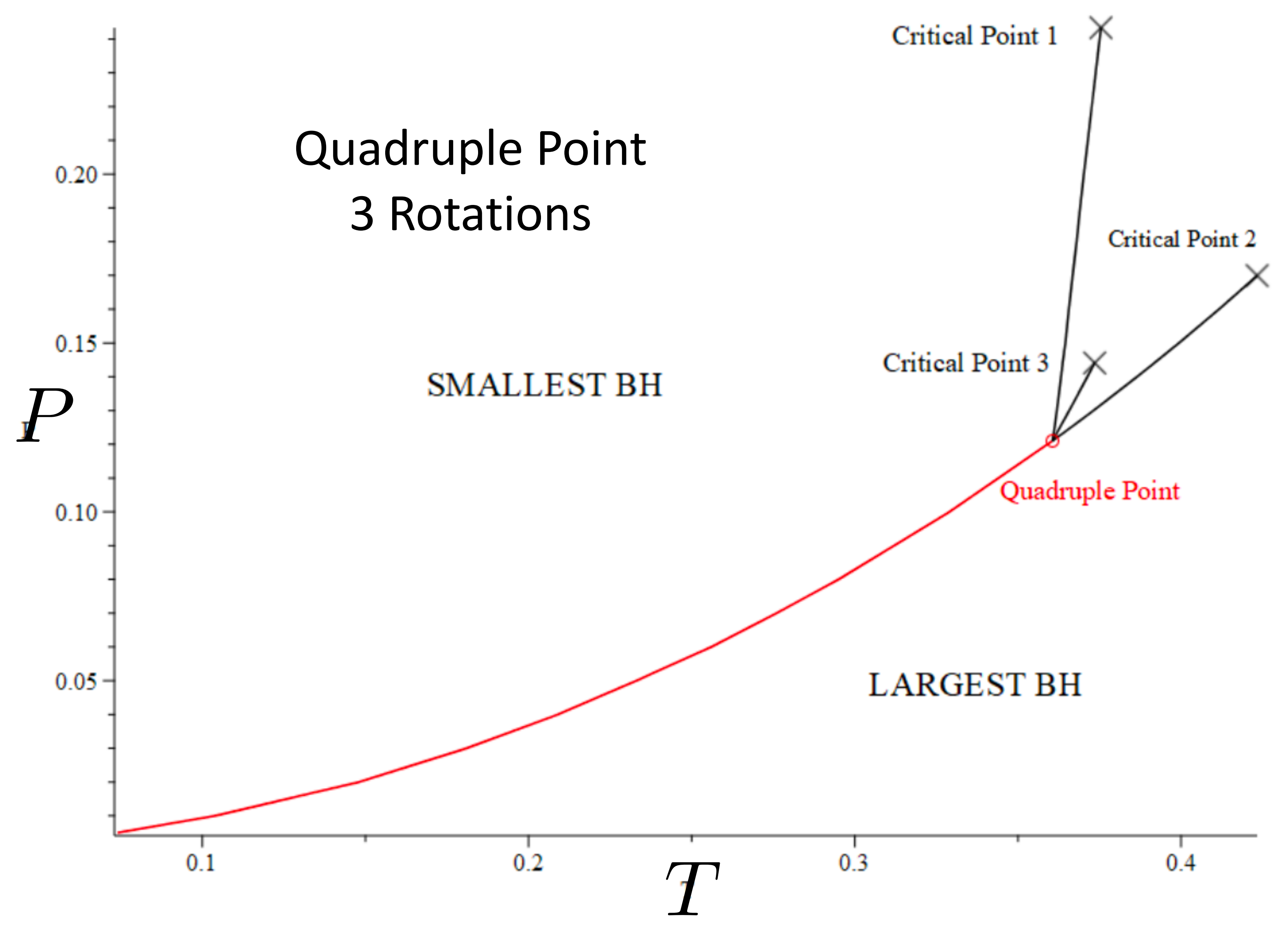
$$T = \frac{1}{2\pi}\left[r_+\left(\frac{r_+^2}{l^2} + 1\right)\sum_{i=1}^{N} \frac{1}{a_i^2 + r_+^2} - \frac{1}{r_+}\left(\frac{1}{2} - \frac{r_+^2}{2l^2}\right)^{\epsilon}\right]$$

$$S = \frac{\omega_{D-2}}{4r_+^{1-\epsilon}}\prod_{i=1}^{N} \frac{a_i^2 + r_+^2}{\Xi_i} \qquad V = \frac{r_+ A}{D-1} + \frac{8\pi}{(D-1)(D-2)}\sum_{i=1}^{n} a_i J_i$$

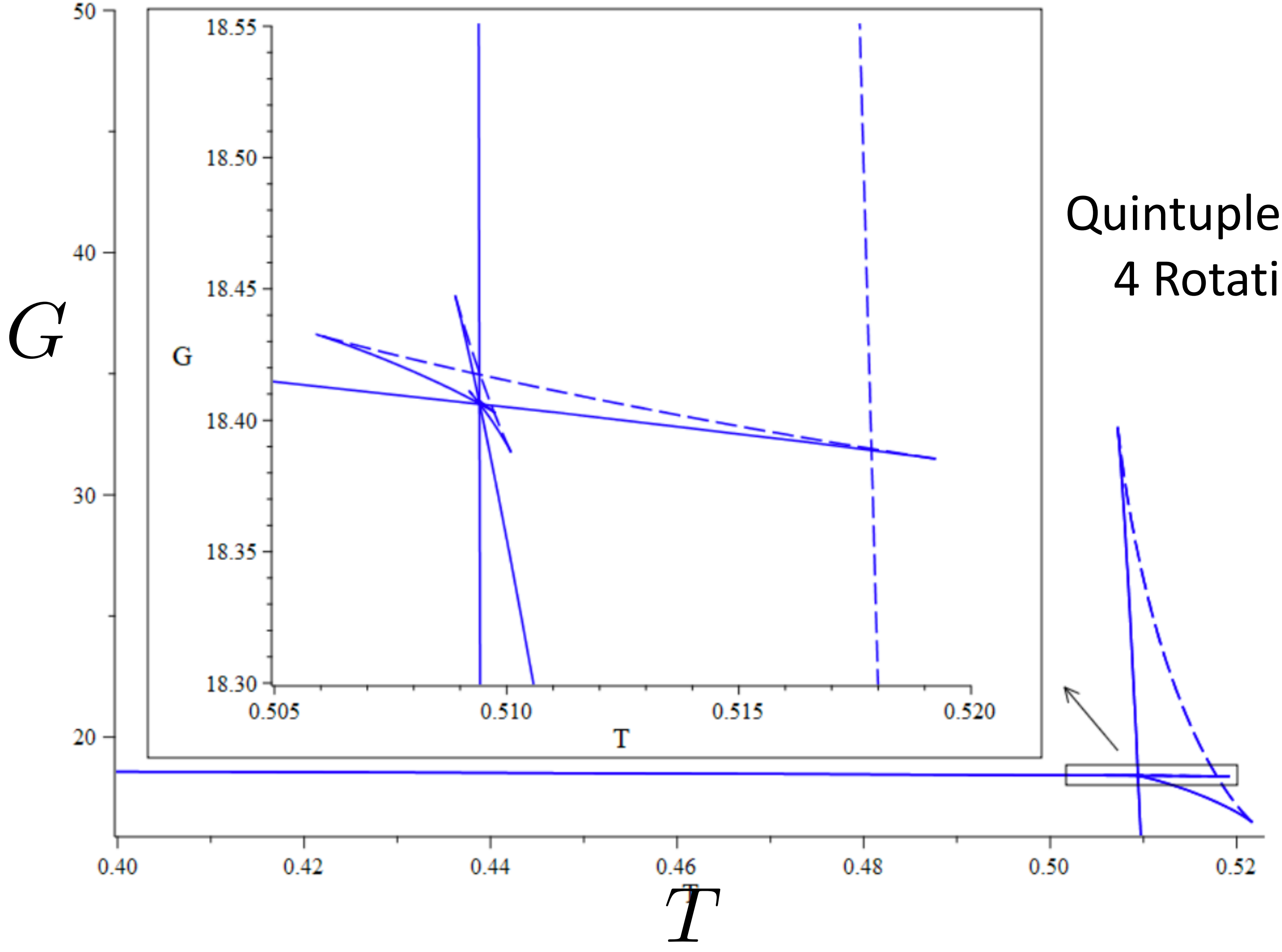
Gibbs Phase Rule

- Two rotations: Triple Point (D > 5)
- Three rotations: Quadruple Point (D > 7)
- Four rotations: Quintuple Point (D > 9)
- Five rotations: Hextuple Point (D > 11)

Quadruple Point
3 Rotations

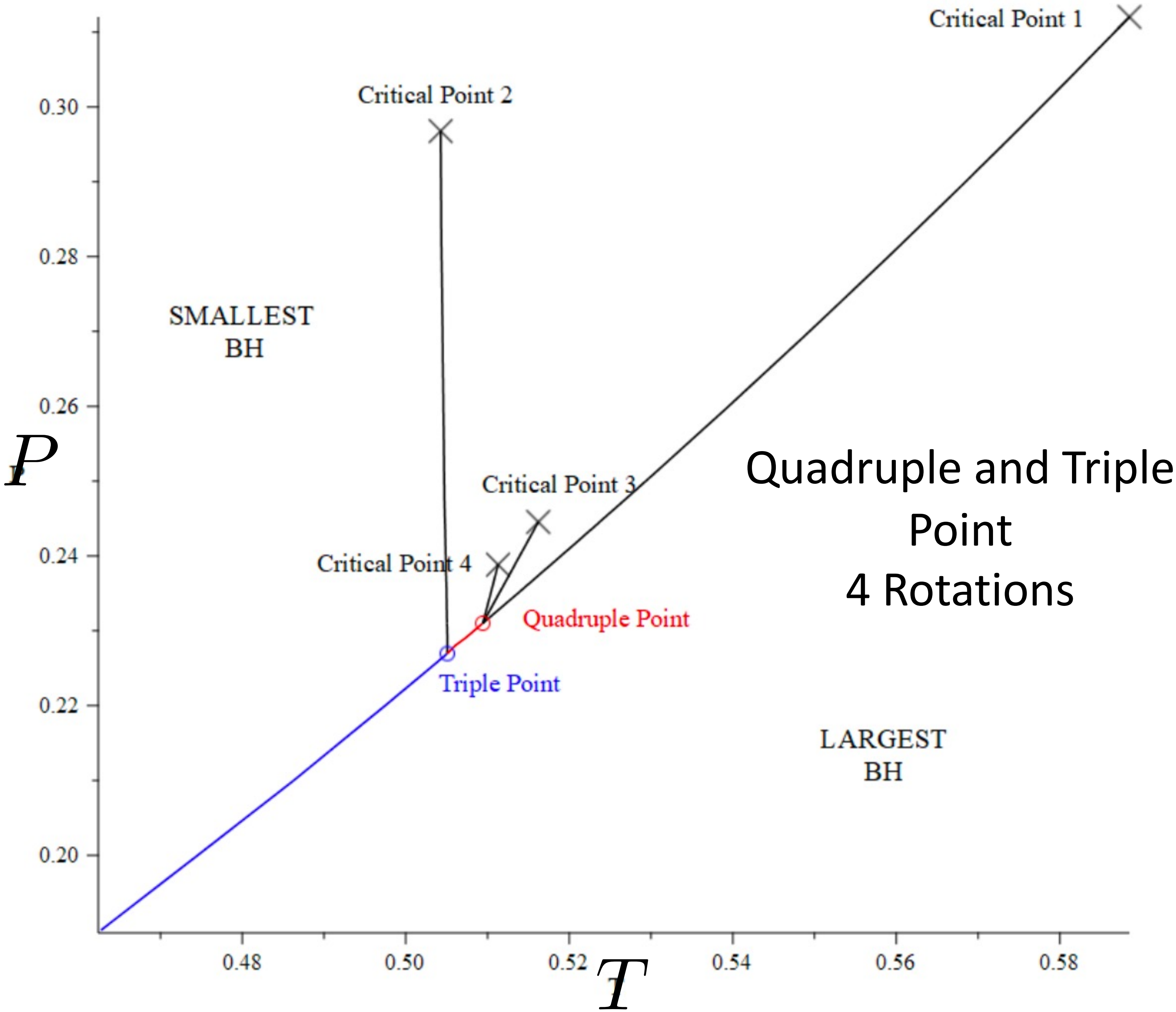


$$D = 8,\ J_1 = 7.967,\ J_2 = 1.24,\ J_3 = 0.12798$$



Quintuple Point
4 Rotations

$$D = 10,\ J_1 = 24.48,\ J_2 = 4.33,\ J_3 = 1.2,\ J_4 = 0.1435$$



Quadruple and Triple Point

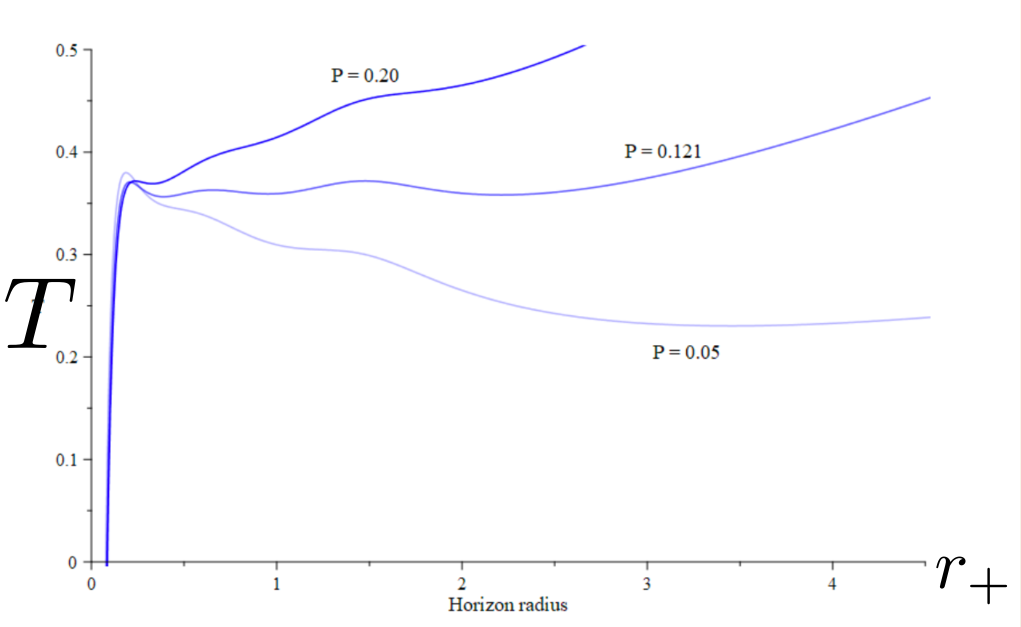
4 Rotations

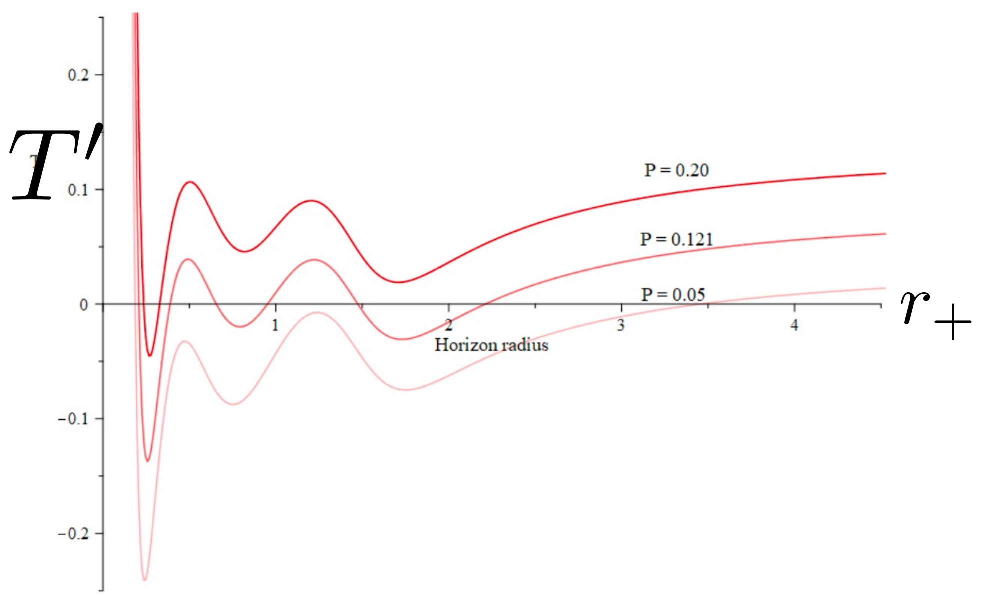
$$d = 10,\ J_1 = 24.48,\ J_2 = 4.331,\ J_3 = 1.1973,\ J_4 = 0.155$$

How?

$$dG = \sum_{i=1}^{N} \Omega_i dJ_i - SdT - VdP$$

- $dG = -SdT$ for constant P and J_i
- extrema of $G(r_+)$ and $T(r_+)$ occur at the same r_+ values



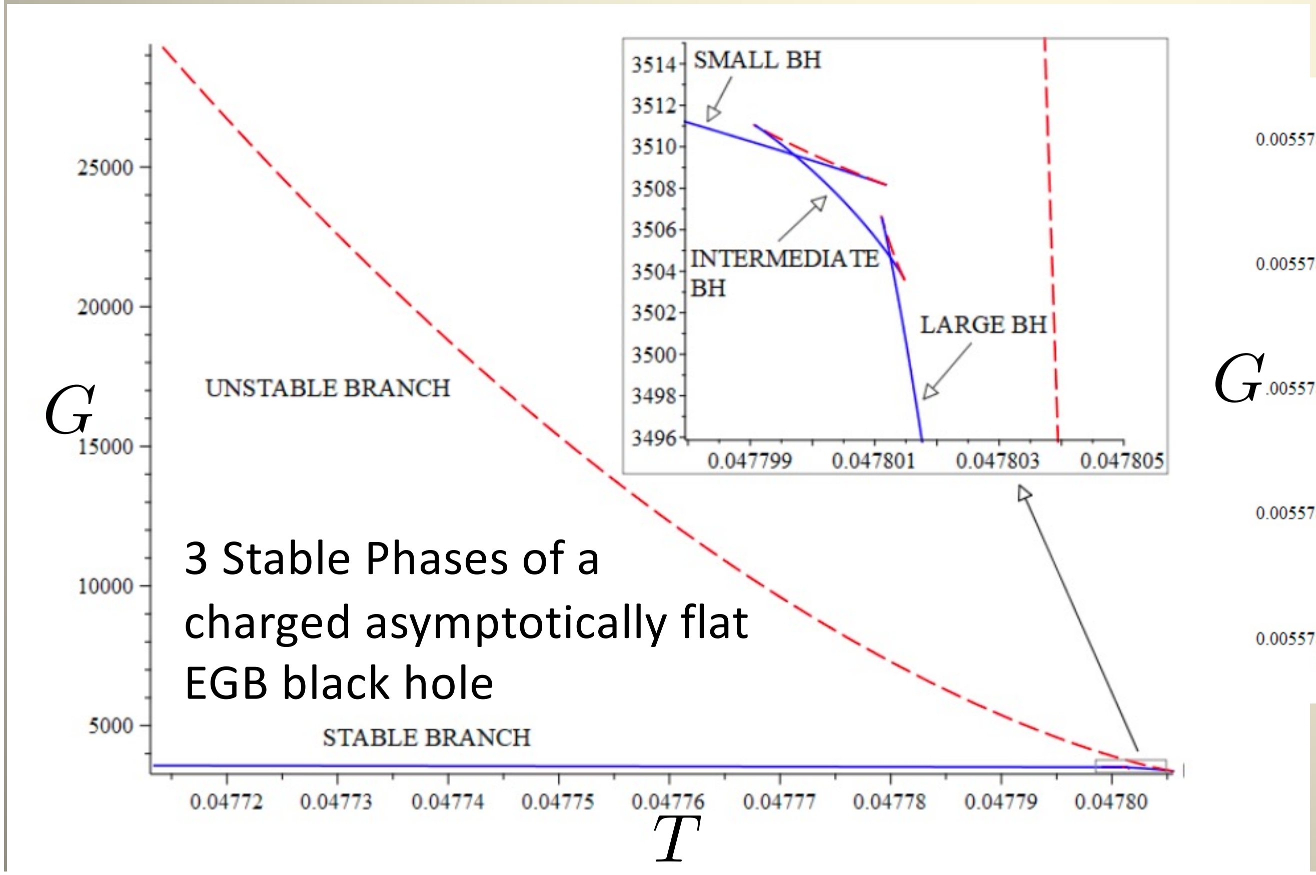


maximum number of coexistent states when $T'(r_+)$ has a root between every local extremum

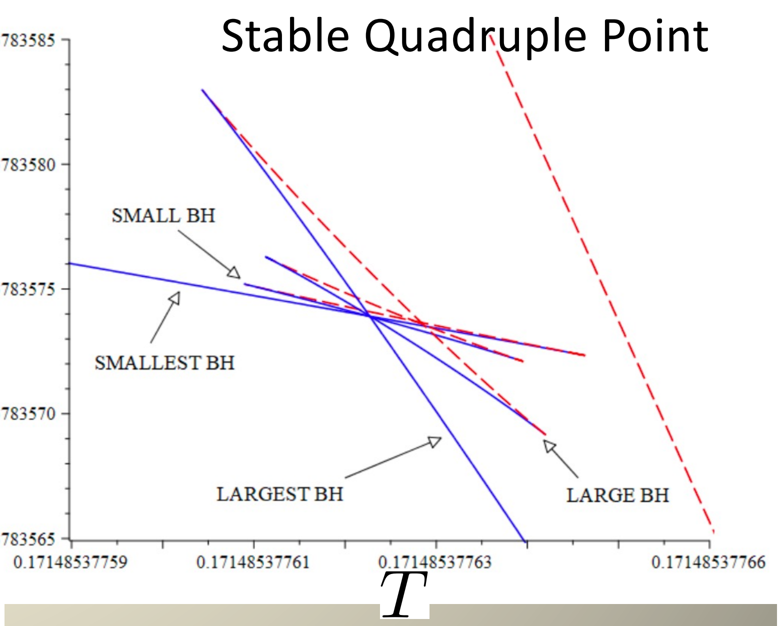
Asymptotically Flat Black Holes

Wu/Mann
CQG (to appear) 2212.08673

- Asymptotically Flat Black holes can exhibit multiple phases and multicritical points!



3 Stable Phases of a charged asymptotically flat EGB black hole


Stable Quadruple Point

Even more Black Hole Chemistry

- Joule-Thompson Expansion

  Okcu/Aydiner
  Eur.Phys.J.C **77** (2017) 1, 24

  - High-to-low pressure changes in black holes mimic those observed in real-world gases
- Thermodynamic Topological Defects

  Wei/Liu/Mann
  PRL **129** (2022) 191101

  - black hole solutions as defects in thermodynamic parameter space
  - winding numbers indicate stability
- Dynamics of Phase Transitions

  Li/Wang PRD **102** (2020) 024085
  Li/Zhang/Wang JHEP **2010** (2020) 09

  - Can use Smoluchowski Eqn to probe. dynamics of phase changes
  - Works at multicritical points

Wei/Wang/Liu/Mann
Sci.Chin.Phys.M.Ast.
64 (2021) 270411
Yang/Mann
2304.08969

Take Home Lesson

- Thermodynamics is Thermodynamics
  - Black Holes exhibit pretty much every behaviour seen in chemistry under the right circumstances
  - Lesson for Quantum Gravity?
- AdS is not a restriction
  - Chemical Phenomena seen in asymptotically flat and asymptotically de Sitter settings
- Holographic understanding emerging
- Still a lot to learn from Black Hole Chemistry

Kubiznak/Mann/Teo,
Black hole chemistry: Thermodynamics with Lambda, CQG **34** (2017) 063001