Black Holes in Einstein-Maxwell, Nonlinear Electrodynamics, and Yang-Mills Theories: Classical and Quantum

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Outline of Lectures

- Einstein-Maxwell theory
- Emission (Hawking radiation and Schwinger effect) from charged BHs
- Nonlinear Electrodynamics (NLED)*
- Einstein-NED BHs*
- Einstein-Yang Mills theory*
- Conclusion

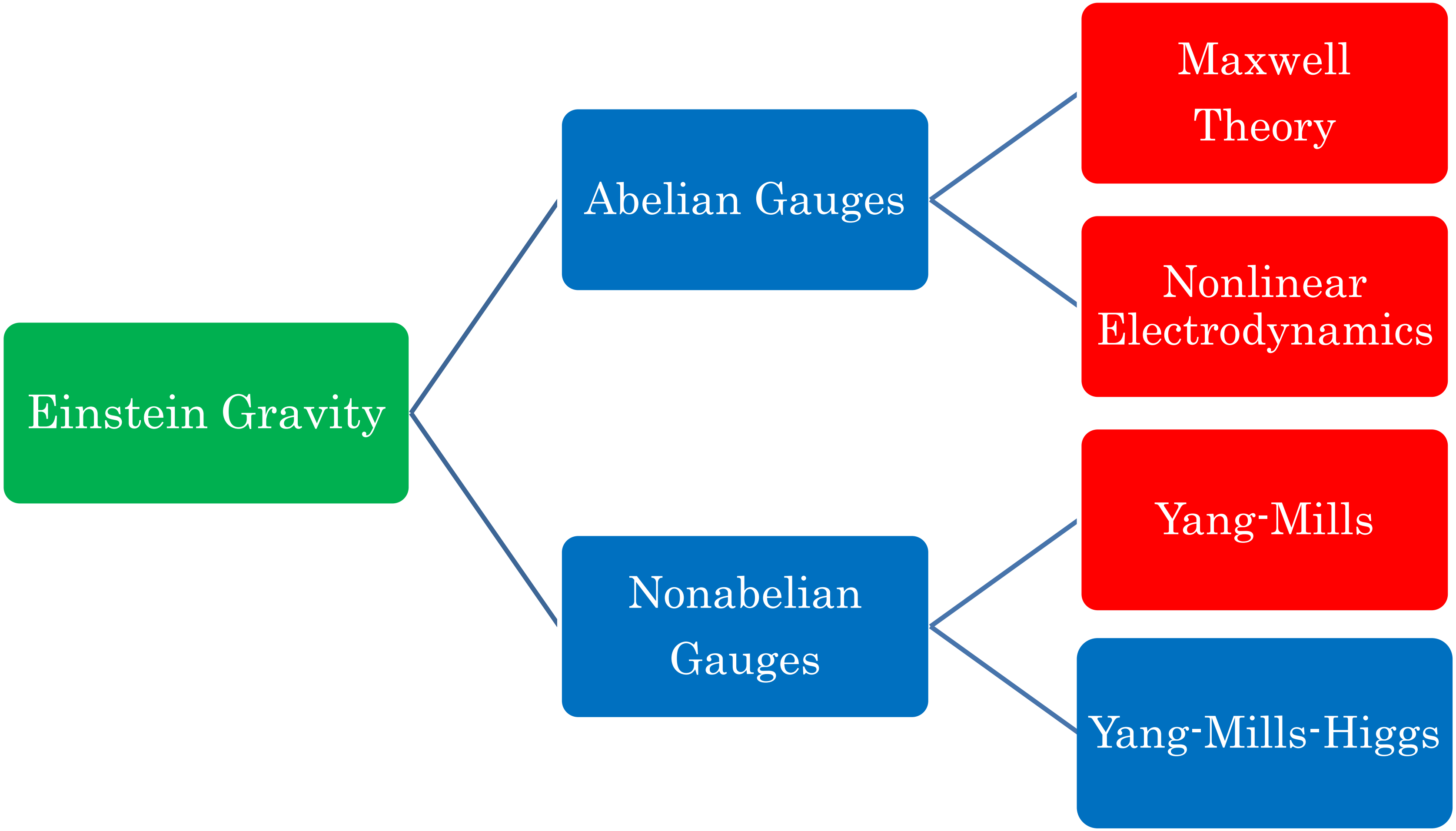
✓ Related to Hong Lu on Einstein-Maxwell theory, in particular, KN BHs and Plebanski BHs, and Robert Mann on BH chemistry & Bin Chen and Ugo Moschella's talks.

✓ Caveat: D = 4 unless mentioned and no backreaction considered.

Why Einstein-Maxwell theory and charged BHs?

- BHs and GWs have been observed (EHT, LIGO-Virgo).
- Magnetic fields (plasma accretion disk) around BHs play important role for central engine for GRBs (Blandford-Znajek) and jets.
- Highly magnetized neutron stars (magnetars) have been observed and are speculated as a possible origin for bursts.
- Magnetars and EM fields during merger of neutron stars go beyond the Maxwell theory ($B \geq B_c$) and need NLED.
- BHs with nonabelian monopoles or dyons might have been formed in the early universe.
- Astrophysics and laboratory astrophysics under extreme conditions (gravity and EM fields).

Einstein-Gauge Theories



Kerr-Newman-(Anti)de Sitter BHs

- Kerr-Newman (KN)-(Ant-)de Sitter ((A)dS) black holes in Boyer-Lindquist-type coordinates ((A)dS radius $L_{(A)dS} = \sqrt{3/|\Lambda|}$)

$$ds^2 = -\frac{\Delta_r}{\varrho^2}\left(dt - \frac{a\sin\theta^2}{\Xi}d\phi\right)^2 + \frac{\varrho^2}{\Delta_r}dr^2 + \frac{\varrho^2}{\Delta_\theta}d\theta^2 + \frac{\Delta_\theta \sin\theta^2}{\varrho^2}\left(adt - \frac{r^2+a^2}{\Xi}d\phi\right)^2$$

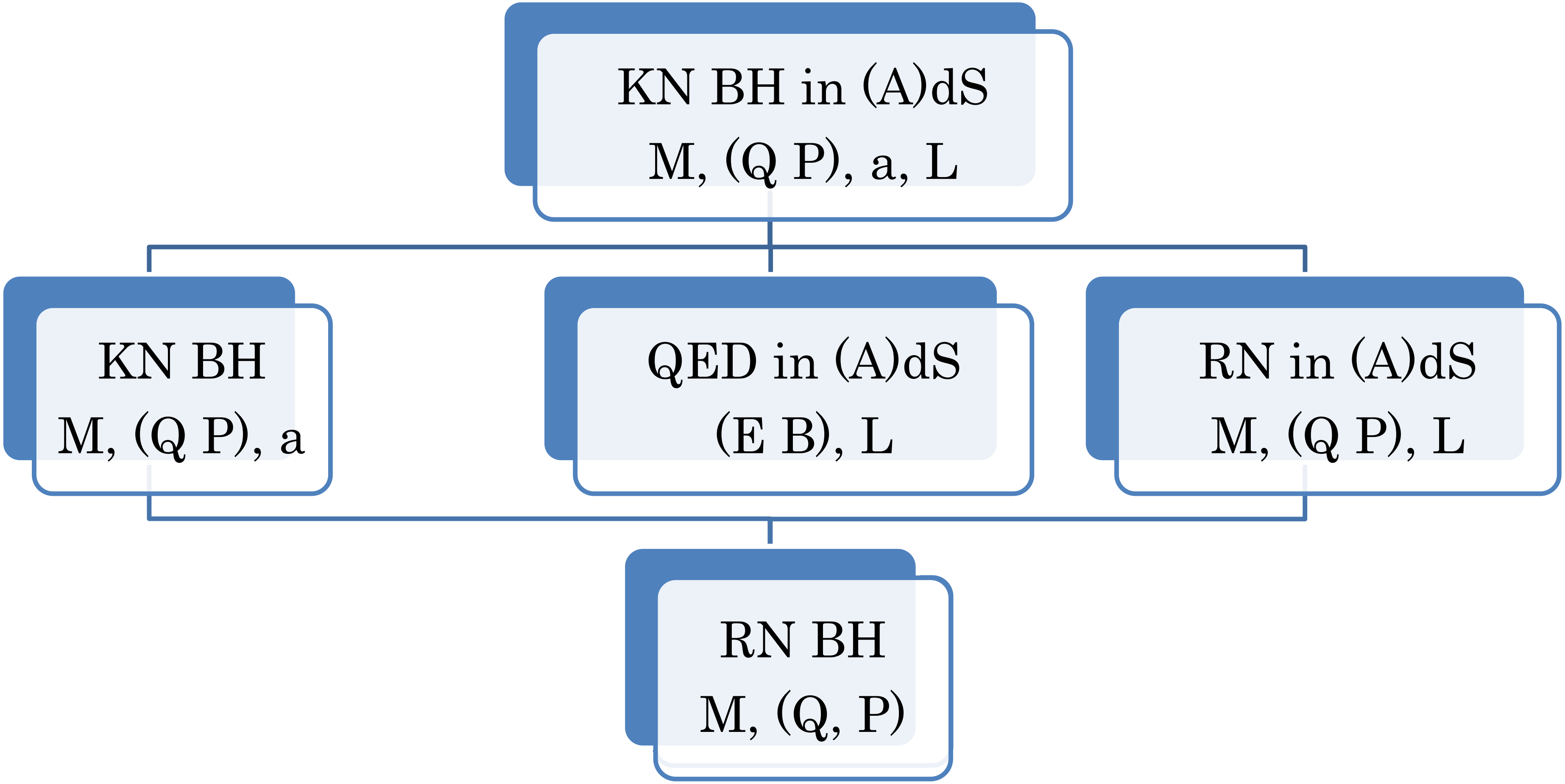
$$\varrho^2 = r^2 + a^2\cos\theta^2\,, \qquad \Delta_\theta = 1 \mp \frac{a^2}{L^2}\cos\theta^2$$

$$\Delta_r = (r^2+a^2)\left(1 \pm \frac{a^2}{L^2}r^2\right) - 2Mr + (Q^2+P^2)\,, \qquad \Xi = 1 \mp \frac{a^2}{L^2}$$

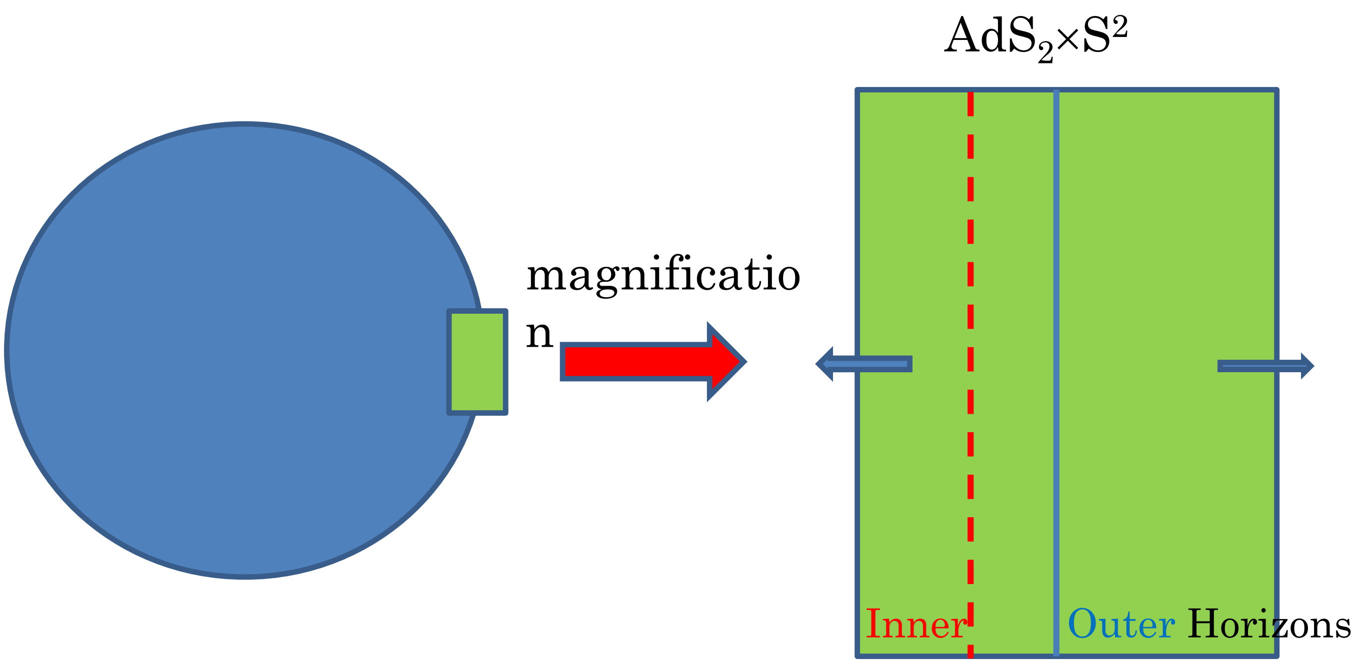
- Mass, angular momentum, electric and magnetic charges: $\frac{M}{\Xi^2}, \frac{Ma}{\Xi^2}, \frac{Q}{\Xi}, \frac{\mathrm{P}}{\Xi}$
- Vector potential for Maxwell field

$$A_{[1]} = \frac{Qr - Pa\cos\theta}{\varrho^2}\left(dt - \frac{a\sin\theta^2}{\Xi}d\phi\right) + \frac{P(\cos\theta - \sigma)}{\Xi}d\phi$$

Black Holes in Einstein-Maxwell Theory in (A)dS Space



Near-horizon Geometry of Near-extremal RN BH



[G. t'Hooft & A. Strominger, "conformal symmetry near the horizon of BH," MG14, July 2015. And Bin Chen's talk]

Near-Horizon Geometries

- **Theorem (Static)** [Kunduri, Lucietti, Reall, CQG 24 ('07)]

Any static near-horizon geometry is locally a warped product of AdS_2, dS_2 or $R^{1,1}$ and H. If H is simply connected, this statement is global. In this case if H is compact and the strong energy conditions holds, it must be the AdS$_2$ case or the direct product $R^{1,1} \times H$.

- **Theorem (Rotational)**

Consider a D-dimensional spacetime containing a degenerate horizon, invariant under an $R \times U(1)^{D-3}$ isometry group, and satisfying the Einstein equations $R_{\mu\nu} = \Lambda g_{\mu\nu}$. Then the near-horizon geometry has a global $G \times U(1)^{D-3}$ symmetry, where G is either $O(2, 1)$ or the 2D Poincare group. Furthermore, if $\Lambda \leq 0$ and the near-horizon geometry is non-static, the Poincare case is excluded.

Schwinger Effect in Dyonic RN BH in (A)dS

- Dyonic RN black hole in (A)dS: (M, Q, P, L)

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2 \,,$$

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2 + P^2}{r^2} \pm \frac{r^2}{L^2} = \pm \frac{(r - r_+)(r - r_-)(r \pm r_C)(r + r_p)}{L^2 r^2}$$

$$A_{[1]} = \frac{Q}{r} dt + P(\cos\theta \mp 1)d\varphi \,, \qquad \bar{A}_{[1]} = \frac{P}{r} dt - Q(\cos\theta \mp 1)d\varphi$$

- Near-extremal RN BH $(r_+ \to r_-)$ leads to near-horizon geometry $AdS_2 \times S^2$: $t = {}^{t}/_{\varepsilon}$, $r = r_0 + \varepsilon\rho$, $r_\pm = r_0 \pm \varepsilon B$, $M = M_0 + (\varepsilon B)^2 R_S/(2R_{AdS}^2)$ and $\varepsilon \to 0$

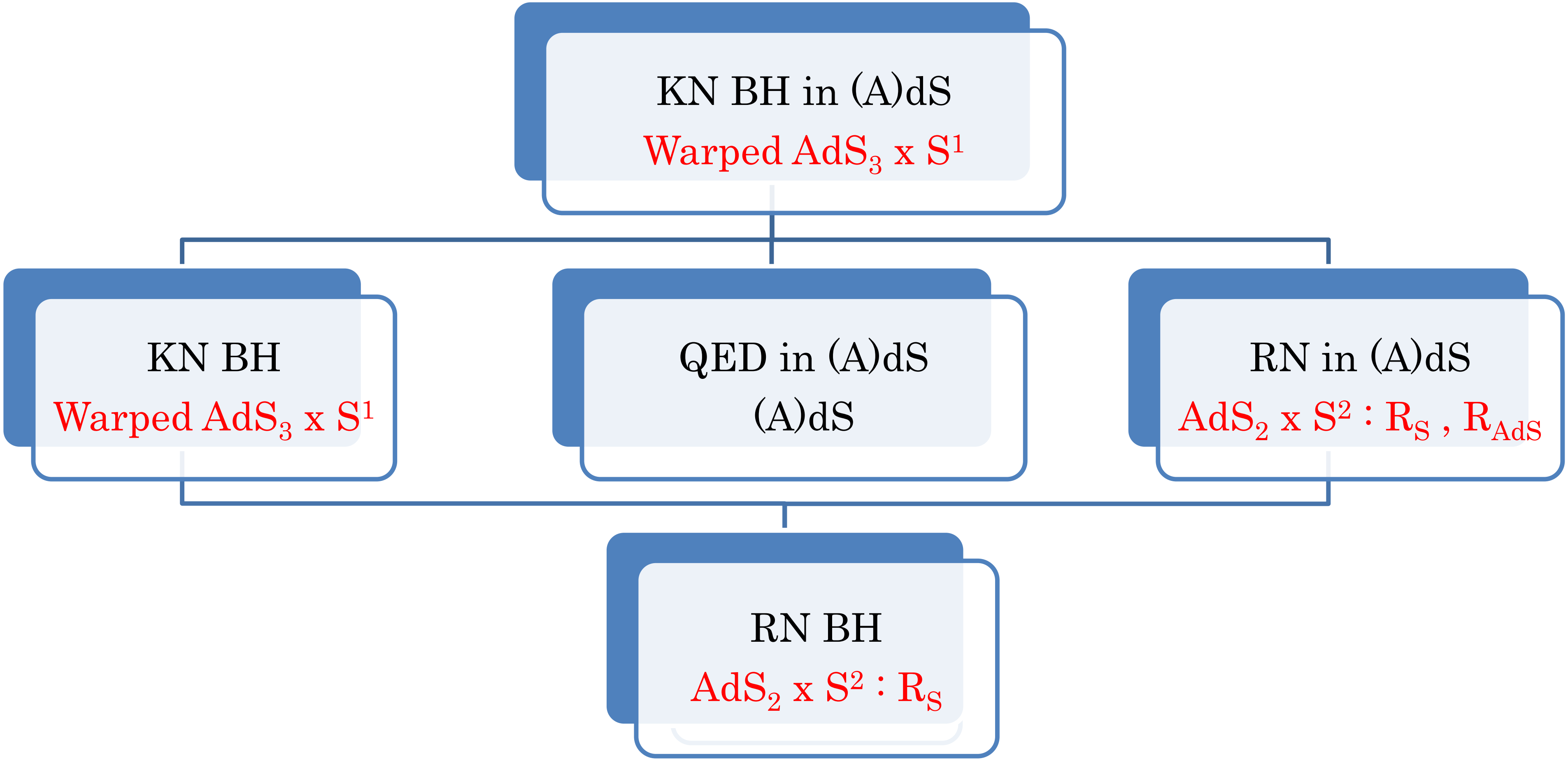
$$ds^2 = -\frac{\rho^2 - B^2}{R_{AdS}^2} d\tau^2 + \frac{R_{AdS}^2}{\rho^2 - B^2} d\rho^2 + R_S^2 d\Omega_2^2, \quad (R_{AdS} \neq R_S \text{ unless } L \neq \infty)$$

$$A_{[1]} = -\frac{Q}{R_S^2} d\tau + P(\cos\theta \mp 1)d\varphi \,,$$

- Hawking temperature and chemical potentials

$$T_H = \frac{B}{2\pi R_{AdS}^2} \,, \qquad \Phi_H = -\frac{QB}{R_S^2} \,, \qquad \bar{\Phi}_H = -\frac{PB}{R_S^2}$$

(Near-) Extremal BHs in (A)dS Space



Petrov Classification of Gravitational Fields

Based on “Exact Space-Times in Einstein’s General Relativity” by Griffiths and Podolsky (Cambrige Monographs, 2009)

Weyl Curvature & Ten Scalar Functions

- Weyl curvature

$$C_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu} + \frac{1}{2}\left(R_{\beta\mu}g_{\alpha\nu} + R_{\alpha\nu}g_{\beta\mu} - R_{\beta\nu}g_{\alpha\mu} - R_{\alpha\mu}g_{\beta\nu}\right) + \frac{1}{6}\left(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}\right)$$

- Metric tensor in null tetrad formalism

$$g_{\mu\nu} = -\left(k_\mu l_\nu + l_\mu k_\nu\right) + m_\mu \bar{m}_\nu + \bar{m}_\mu m_\nu$$

- Ten complex scalar functions for the Weyl curvature

$\Psi_0 = C_{\alpha\beta\mu\nu}k^\alpha m^\beta k^\mu m^\nu$ — A transverse component propagating in $\boldsymbol{l}$ direction

$\Psi_1 = C_{\alpha\beta\mu\nu}k^\alpha l^\beta k^\mu m^\nu$ — A longitudinal component propagating in $\boldsymbol{l}$ direction

$\Psi_2 = C_{\alpha\beta\mu\nu}k^\alpha m^\beta \bar{m}^\mu l^\nu$ — A Coulomb-like component

$\Psi_3 = C_{\alpha\beta\mu\nu}l^\alpha k^\beta l^\mu \bar{m}^\nu$ — A longitudinal component propagating in $\boldsymbol{k}$ direction

$\Psi_4 = C_{\alpha\beta\mu\nu}l^\alpha \bar{m}^\beta l^\mu \bar{m}^\nu$ — A transverse component propagating in $\boldsymbol{k}$ direction

Petrov Classification

Type	Multiplicity of eigenvalues	Weyl curvature	Zero scalars	Nonzero scalar
I	[1,1,1,1] (algebraically general)	$k_{[\delta}C_{\alpha]\beta\mu[\nu}k_{\sigma]}k^{\beta}k^{\mu}=0$	$\Psi_0 = 0$	$\Psi_1 \neq 0$
II	[2,1,1] (algebraically special)	$C_{\alpha\beta\mu[\nu}k_{\gamma]}k^{\beta}k^{\mu}=0$	$\Psi_0 = \Psi_1 = 0$	$\Psi_2 \neq 0$
D	[2,2] (algebraically special)		$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$	$\Psi_2 \neq 0$
III	[3,1] (algebraically special)	$C_{\alpha\beta\mu[\nu}k_{\gamma]}k^{\mu}=0$	$\Psi_0 = \Psi_1 = \Psi_2 = 0$	$\Psi_3 \neq 0$
N	[4] (algebraically special)	$C_{\alpha\beta\mu\nu}k^{\mu} = 0$	$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0$	$\Psi_4 \neq 0$
O	Conformally flat			

Einstein-Maxwell Theory

Based on “Exact Space-Times in Einstein’s General Relativity” by Griffiths and Podolsky (Cambrige Monographs, 2009)

Einstein Equation

- Einstein field equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} \ , \qquad \left(G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right)$$

- Scalar functions for Ricci tensor

$$\Phi_{00} = \frac{1}{2} R_{\mu\nu} k^{\mu} k^{\nu} \ , \qquad \Phi_{22} = \frac{1}{2} R_{\mu\nu} l^{\mu} l^{\nu}$$

$$\Phi_{01} = \frac{1}{2} R_{\mu\nu} k^{\mu} m^{\nu} \ , \qquad \Phi_{12} = \frac{1}{2} R_{\mu\nu} l^{\mu} m^{\nu}$$

$$\Phi_{02} = \frac{1}{2} R_{\mu\nu} m^{\mu} m^{\nu} \ , \qquad \Phi_{11} = \frac{1}{4} R_{\mu\nu} (k^{\mu} l^{\nu} + m^{\mu} \bar{m}^{\nu})$$

Maxwell Theory

- Electromagnetic field:
  - Maxwell tensor $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$; differential forms $\boldsymbol{F} = d\boldsymbol{A}$
  - Dual tensor $\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$, $\quad (\varepsilon_{0123} = \sqrt{-g})$
- Maxwell equations

$$F^{\mu\nu}{}_{;\nu} = 4\pi J^{\mu}\,, \qquad \tilde{F}^{\mu\nu}{}_{;\nu} = 0$$

- Energy-momentum tensor

$$T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\alpha}F_{\nu}^{\alpha} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right)$$

- Complex scalar functions

$$\Phi_0 = F_{\mu\nu}k^{\mu}m^{\nu}\,, \qquad \Phi_1 = \frac{1}{2}F_{\mu\nu}(k^{\mu}l^{\nu} + \bar{m}^{\mu}m^{\nu})\,, \qquad \Phi_2 = \bar{m}^{\mu}m^{\nu}$$

- Einstein electrovacuum solution $(J = 0)$

$$\Phi_{AB} = 2\Phi_A\bar{\Phi}_B$$

de Sitter (dS) Space

Maximally symmetric space with constant curvature R:

$$R_{\alpha\beta\mu\nu} = \frac{1}{12} R\left(g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu}\right)$$

$$R = D(D-1)H^2 = \frac{2D}{D-2}\Lambda, \qquad \left(L_d = \frac{1}{H}\right)$$

dS Space

- Globally embedded hyperboloid in D=5

$$-Z_0^2 + Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2 = L_d^2 , \qquad \left(L_d = \sqrt{3/\Lambda} = 1/H\right)$$

$$ds^2 = -dZ_0^2 + dZ_1^2 + dZ_2^2 + dZ_3^2 + dZ_4^2$$

- Global coordinates

$$Z_0 = \frac{1}{H}\sinh Ht \,, Z_1 = \frac{1}{H}\cosh Ht \cos\chi \,, Z_2 = \frac{1}{H}\cosh Ht \sin\chi\cos\theta$$

$$Z_3 = \frac{1}{H}\cosh Ht \sin\chi \sin\theta\cos\phi \,, Z_4 = \frac{1}{H}\cosh Ht \sin\chi\cos\theta\sin\phi$$

- Friedmann-Lemaitre-Robertson-Walker (FLRW) geometry (spatial curvature k=+1)

$$ds^2 = -dt^2 + \left(\frac{\cosh Ht}{H}\right)^2 \left(d\chi^2 + \sin\chi^2\left(d\theta^2 + \sin\theta^2\, d\phi^2\right)\right)$$

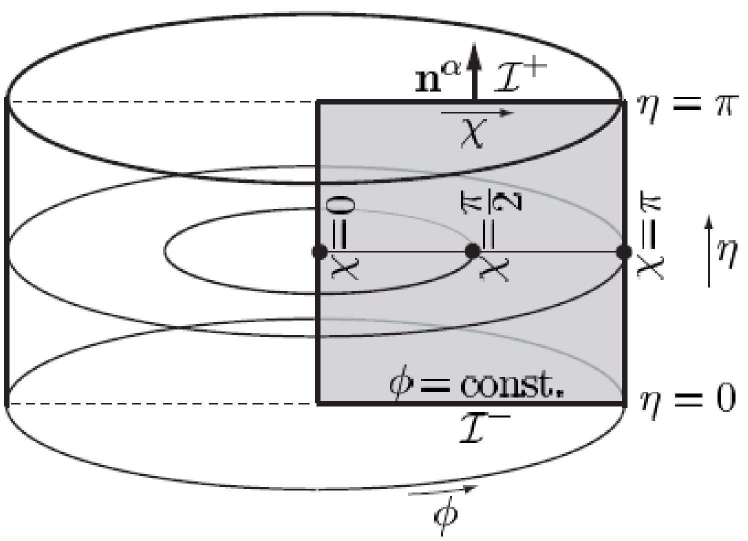
dS Space

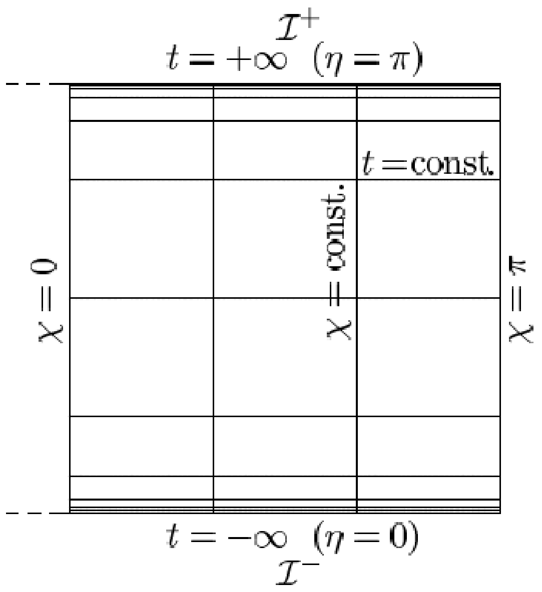
- Conformal time

$$\sin\eta = \operatorname{sech} Ht\,, \qquad (\tan\eta/2 = e^{Ht})$$

$$ds^2 = \left(\frac{1}{H\sin\eta}\right)^2 \left(-d\eta^2 + d\chi^2 + \sin\chi^2\,(d\theta^2 + \sin\theta^2\, d\phi^2)\right)$$

- Conformal structure and Penrose diagram





dS Space

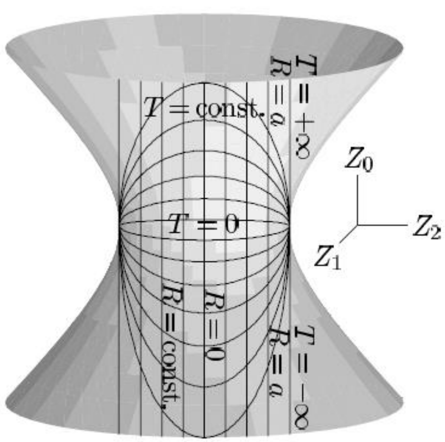
- Spherical coordinates

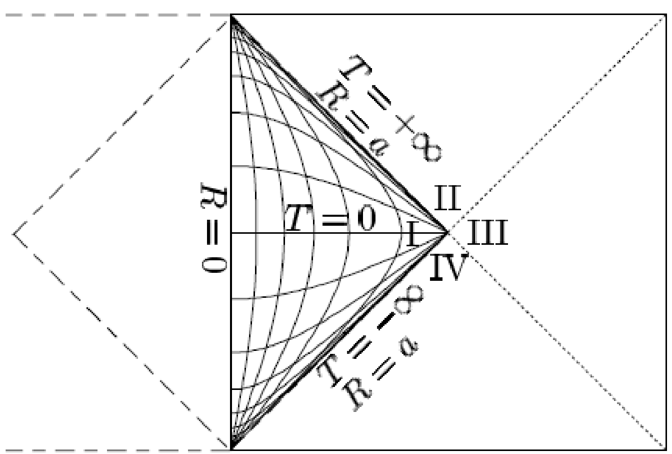
$$Z_0 = \sqrt{L_d^2 - r^2}\sinh\frac{t}{L_d}, \qquad Z_1 = \sqrt{L_d^2 - r^2}\cosh\frac{t}{L_d},$$

$$Z_2 = r\cos\theta\,, \qquad Z_3 = r\sin\theta\cos\phi\,, \qquad Z_4 = r\cos\theta\sin\phi$$

- Metric

$$ds^2 = -\left(1 - \frac{r^2}{L_d^2}\right)dt^2 + \frac{dr^2}{1 - \frac{r^2}{L_d^2}} + r^2(d\theta^2 + \sin\theta^2\, d\phi^2)$$





dS Space

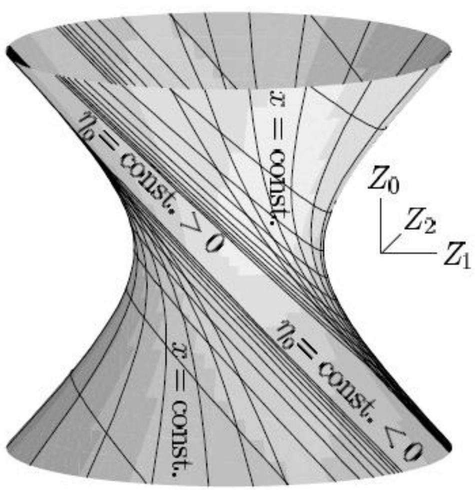
- Conformally flat coordinates ($s = -\eta_0^2 + x^2 + y^2 + z^2$)

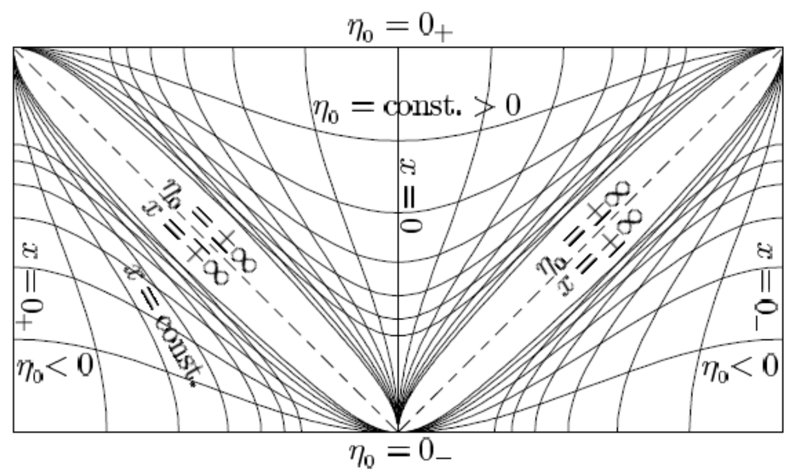
$$Z_0 = \frac{L_d^2 + s}{2\eta_0}, Z_1 = \frac{L_d^2 - s}{2\eta_0}, Z_2 = L_d \frac{x}{\eta_0}, Z_3 = L_d \frac{y}{\eta_0}, Z_4 = L_d \frac{z}{\eta_0}$$

- Conformally flat metric

$$ds^2 = \frac{L_d^2}{\eta_0^2}(-d\eta_0^2 + dx^2 + dy^2 + dz^2)$$

- FLRW geometry in planar coordinates $\eta_0 = e^{-Ht}/H$





Anti-de Sitter (AdS) Space

Maximally symmetric space with constant curvature R:

$$R_{\alpha\beta\mu\nu} = \frac{1}{12} R\left(g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu}\right)$$

$$R = -D(D-1)K^2 = \frac{2D}{D-2}\Lambda, \qquad \left(L_a = \frac{1}{K}\right)$$

AdS Space

- Globally embedded hyperboloid in D=5

$$-Z_0^2 + Z_1^2 + Z_2^2 + Z_3^2 - Z_4^2 = L_a^2 , \qquad \left(L_a = \sqrt{-3/\Lambda} = 1/K \right)$$

$$ds^2 = -dZ_0^2 + dZ_1^2 + dZ_2^2 + dZ_3^2 - dZ_4^2$$

- Global coordinates

$$Z_0 = \frac{1}{K} \cosh r \sin Kt \, , Z_1 = \frac{1}{K} \sinh r \cos \theta \, , Z_2 = \frac{1}{K} \sinh r \sin \theta \cos \phi$$

$$Z_3 = \frac{1}{K} \sinh r \sin \theta \sin \phi \, , Z_4 = \frac{1}{K} \cosh r \;\cos Kt$$

$$ds^2 = - \cosh r^2 \, dt^2 + \left(\frac{1}{K} \right)^2 \left(dr^2 + \sinh r^2 \left(d\theta^2 + \sin \theta^2 \, d\phi^2 \right) \right)$$

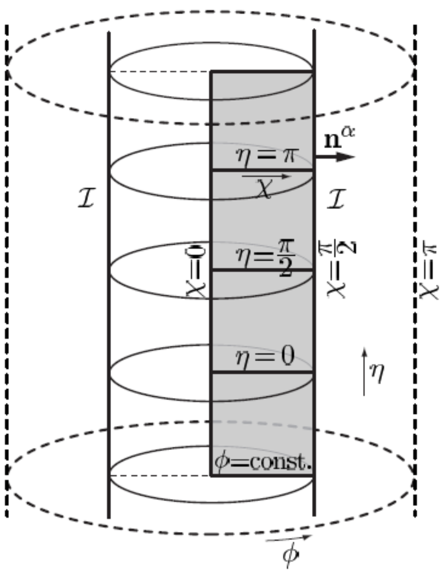
AdS Space

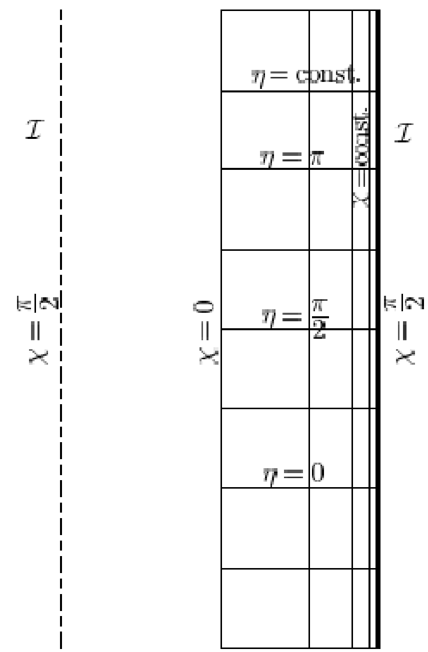
- Conformal coordinate and time

$$\tan\chi = \sinh r \;, \qquad \eta = Kt$$

$$ds^2 = \left(\frac{1}{K\cos\chi}\right)^2 \left(-d\eta^2 + d\chi^2 + \sin\chi^2\left(d\theta^2 + \sin\theta^2\, d\phi^2\right)\right)$$

- Conformal structure and Penrose diagram





AdS Space

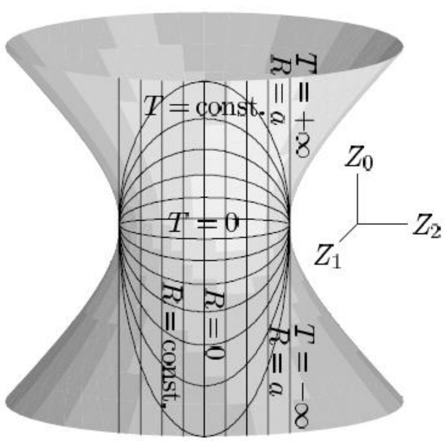
- Spherical coordinates

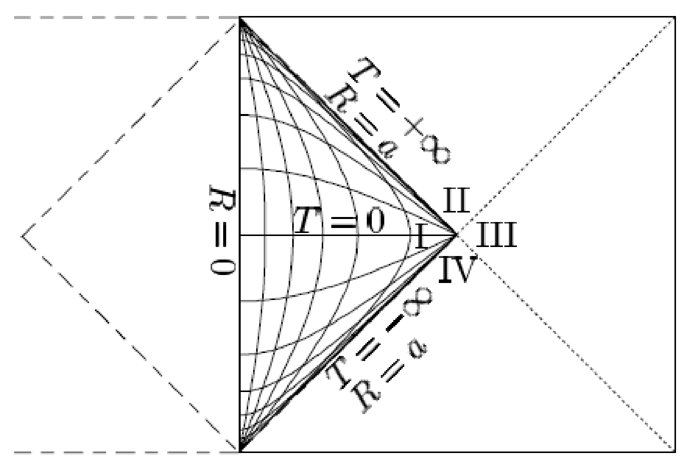
$$Z_0 = \sqrt{L_a^2 + r^2}\sin\frac{t}{L_a}, \qquad Z_4 = \sqrt{L_a^2 + r^2}\cos\frac{t}{L_a},$$

$$Z_1 = r\cos\theta\,, \qquad Z_2 = r\sin\theta\cos\phi\,, \qquad Z_4 = r\cos\theta\sin\phi$$

- Metric

$$ds^2 = -\left(1 + \frac{r^2}{L_a^2}\right)dt^2 + \frac{dr^2}{1 + \frac{r^2}{L_a^2}} + r^2(d\theta^2 + \sin\theta^2\, d\phi^2)$$



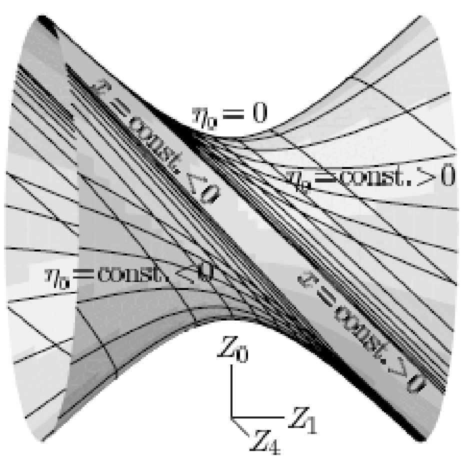


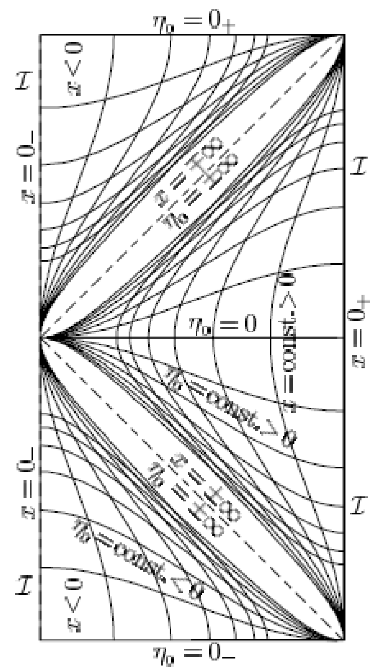
AdS Space

- Conformally flat coordinates ($s = -\eta_0^2 + x^2 + y^2 + z^2$)

$$Z_0 = \frac{L_a^2 + s}{2x}, Z_1 = \frac{L_d^2 - s}{2x}, Z_2 = L_a \frac{y}{x}, Z_3 = L_a \frac{z}{x}, Z_4 = L_a \frac{\eta_0}{x}$$

- Conformally flat metric: $ds^2 = \frac{L_a^2}{x^2}(-d\eta_0^2 + dx^2 + dy^2 + dz^2)$
- Planar coordinates ($x = \pm \frac{e^{K\hat{x}}}{K}$) : $ds^2 = d\hat{x}^2 + e^{2K\hat{x}}(-d\eta_0^2 + dy^2 + dz^2)$





Electrovacuum Solutions

Based on “Exact Space-Times in Einstein’s General Relativity”
by Griffiths and Podolsky (Cambrige Monographs, 2009)

Bertotti-Robinson Universe

- Conformally flat solution of Einstein-Maxwell equations for non-null EM field [Bertotti ('59), Robinson ('59)]

$$ds^2 = \frac{q^2}{r^2}\left(-dt^2 + dr^2 + r^2(d\theta^2 + \sin\theta^2\, d\phi^2)\right)$$

$$F = \frac{q}{r^2} dr \wedge dt$$

- Non-null EM field $F_{\mu\nu}F^{\mu\nu} = -2/q^2$ (non-singular spherically symmetric and homogeneous)
- Null tetrad

$$k = \left(\frac{r}{q}\right)\frac{(\partial_t + \partial_r)}{\sqrt{2}}, \qquad l = \left(\frac{r}{q}\right)\frac{(\partial_t - \partial_r)}{\sqrt{2}}$$

- Complex scalar function $\Phi_0 = 1/2q$

Bertotti-Robinson Universe

- Other forms of Bertotti-Robinson universe from transformation

$$t = \frac{q\sqrt{q^2 + R^2}\sin(\frac{T}{q})}{R + \sqrt{q^2 + R^2}\cos(\frac{T}{q})}, \qquad r = \frac{q^2}{R + \sqrt{q^2 + R^2}\cos(\frac{T}{q})}$$

- Metric

$$ds^2 = -\left(1 + \frac{R^2}{q^2}\right) dT^2 + \frac{dR^2}{1 + \frac{R^2}{q^2}} + q^2(d\theta^2 + \sin\theta^2\, d\phi^2)$$

- Topology is $AdS_2 \times S^2$ $(R_{Ads} \neq R_S\ unless\ q = 1)$

Bertotti-Robinson Universe

- Alternative form of Bertotti-Robinson universe from transformation

$$t = q\left(\frac{q+v}{q-v} - \frac{2q-u}{2q+u}\right), \qquad r = q\left(\frac{q+v}{q-v} + \frac{2q-u}{2q+u}\right)$$

$$\zeta = \sqrt{2}q\tan\theta/2\ e^{i\phi}$$

- Metric

$$ds^2 = -\frac{2dudv}{(1+uv/2q^2)^2} + \frac{2d\zeta d\bar{\zeta}}{(1+\zeta\bar{\zeta}/2q^2)^2}$$

- A particular subclass of the Kundt family of nonexpanding spacetimes with $u = -q^2/\tilde{u},\ v = 2(\tilde{u} - 2q^2/\tilde{v})$

$$ds^2 = -2d\tilde{u}d\tilde{v} - \frac{\tilde{v}^2}{q^2}d\tilde{u}^2 + \frac{2d\zeta d\bar{\zeta}}{(1+\zeta\bar{\zeta}/2q^2)^2}$$

Direct-product Spacetimes

- Parametrized metric for all direct-product spacetimes

$$ds^2 = -\frac{2dudv}{(1-\epsilon_1 uv/2a^2)^2} + \frac{2d\zeta d\bar{\zeta}}{(1+\epsilon_2 \zeta\bar{\zeta}/2b^2)^2}$$

- Null tetrad: $k = \Omega\partial_v\,, l = \Omega\partial_u\,, m = \Sigma\partial_{\bar{\zeta}}\,, \bar{m} = \Sigma\partial_\zeta$ ($\Omega = 1 - \epsilon_1 uv/2a^2, \Sigma = 1 + \epsilon_2\zeta\bar{\zeta}/2b^2$)
- Nonvanishing scalars (type D)

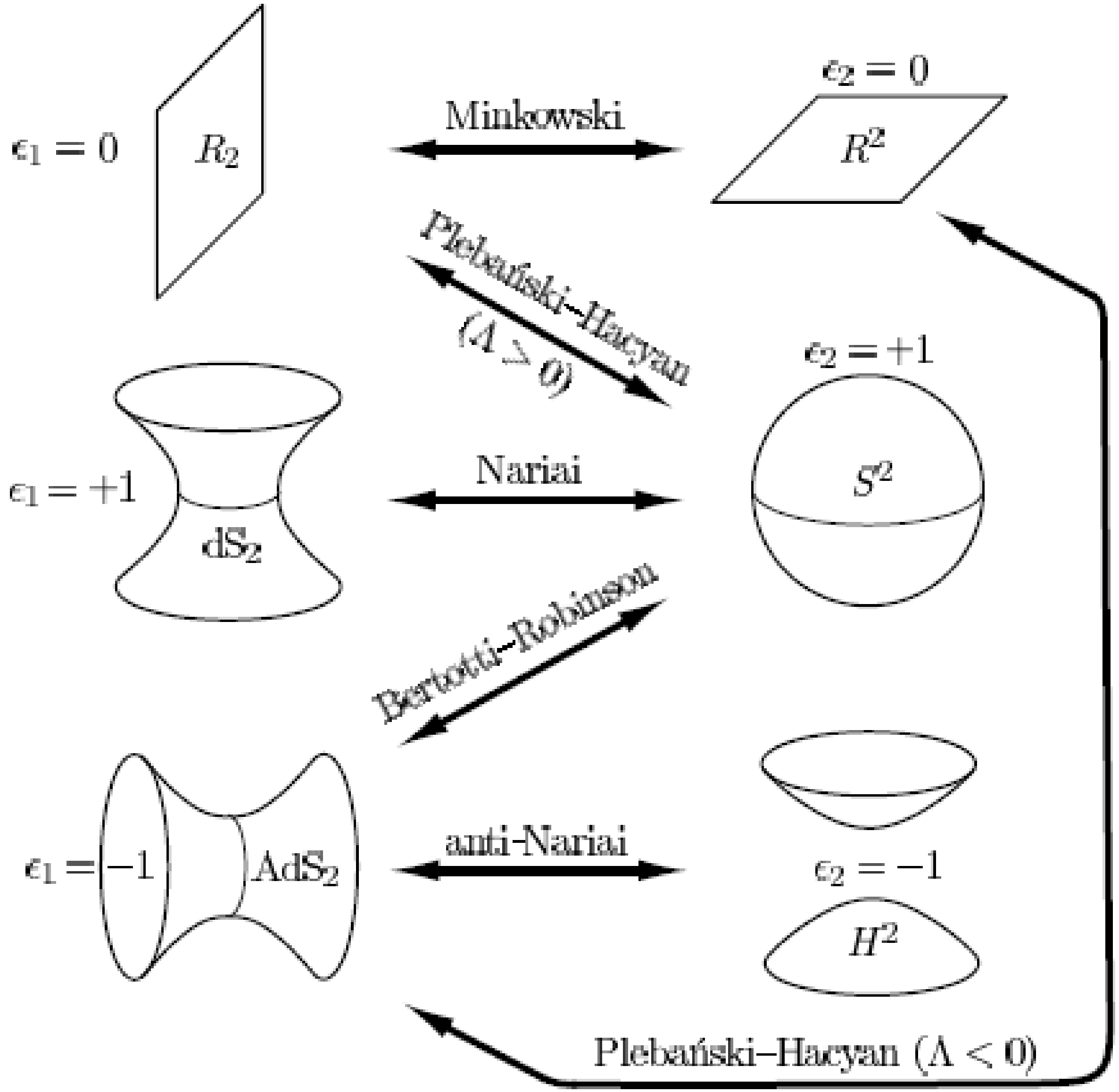
$$\Psi_2 = -\frac{1}{6}\left(\frac{\epsilon_1}{a^2} + \frac{\epsilon_2}{b^2}\right), \qquad \Phi_{11} = \frac{1}{4}\left(-\frac{\epsilon_1}{a^2} + \frac{\epsilon_2}{b^2}\right), \qquad R = \frac{1}{2}\left(\frac{\epsilon_1}{a^2} + \frac{\epsilon_2}{b^2}\right)$$

- Classification

Direct-product Spacetimes

ϵ_1	ϵ_2	Geometry	Universe	Φ_{11}	Λ
0	0	$M_2 \times E^2$	Minkowski	$= 0$	$= 0$
+1	+1	$dS_2 \times S^2$	Nariai	$= 0$	> 0
-1	-1	$AdS_2 \times H^2$	Anti-Nariai	$= 0$	< 0
-1	+1	$AdS_2 \times S^2$	Bertotti-Robinson	> 0	$= 0$
0	+1	$M_2 \times S^2$	Plebanski-Hacyan	> 0	> 0
-1	0	$AdS_2 \times E^2$	Plebanski-Hacyan	> 0	< 0

Direct-product Spacetimes



Melvin Universe

- Cylindrically symmetric metric [Bonner ('54); Melvin ('64)]

$$ds^2 = D^2(\rho)(-dt^2 + d\rho^2 + dz^2) + \frac{1}{D^2(\rho)}\rho^2 d\phi^2$$

$$D(\varrho) = 1 + B^2\varrho^2/4$$

- Complex self-dual Maxwell tensor (non-null EM field)

$$F + i\tilde{F} = e^{-i\psi}B(dz \wedge dt + iD^{-2}(\rho)\rho d\rho \wedge d\phi)$$

  - Electric field ($\psi = 0$): $F = Bdz \wedge dt$
  - Magnetic field ($\psi = \pi/2$): $F = BD^{-2}(\rho)\rho d\rho \wedge d\phi$

- Tetrad

$$k = \frac{1}{\sqrt{2}D}(\partial_t + \partial_z), l = \frac{1}{\sqrt{2}D}(\partial_t - \partial_z), m = \frac{1}{\sqrt{2}}\left(\frac{1}{D}\partial_\rho - i\frac{D}{\rho}\partial_\phi\right)$$

- Nonvanishing components (type D)

$$\Psi_2 = \frac{1}{2}B^2D^{-4}\left(-1 + \frac{1}{4}B^2\varrho^2\right), \Phi_{11} = \frac{1}{2}B^2D^{-4}, \Phi_1 = \frac{1}{2}BD^{-2}e^{i\psi}$$

Black Holes in Einstein-Maxwell Theory

Reissner-Nordström BHs

- Spherically symmetric black holes with mass M, electric charge Q and magnetic charge P

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin\theta^2\, d\phi^2)$$

$$f(r) = 1 - \frac{2m}{r} + \frac{q^2 + p^2}{r^2}$$

- Vector potential for Maxwell field

$$A_{[1]} = \frac{q}{r}dt + p(\cos\theta - \sigma)d\phi$$

- Nonvanishing components of curvatures (type D)

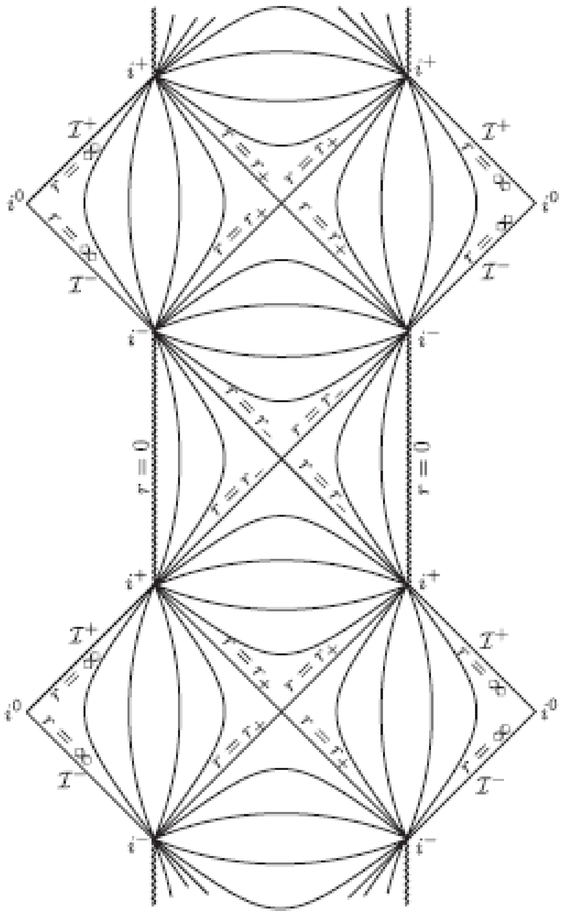
$$\Psi_2 = -\frac{m}{r^3} + \frac{q^2 + p^2}{r^4}, \qquad \Phi_{11} = \frac{q^2 + p^2}{2r^4}$$

- Outer (event) horizon and inner (Cauchy) horizon

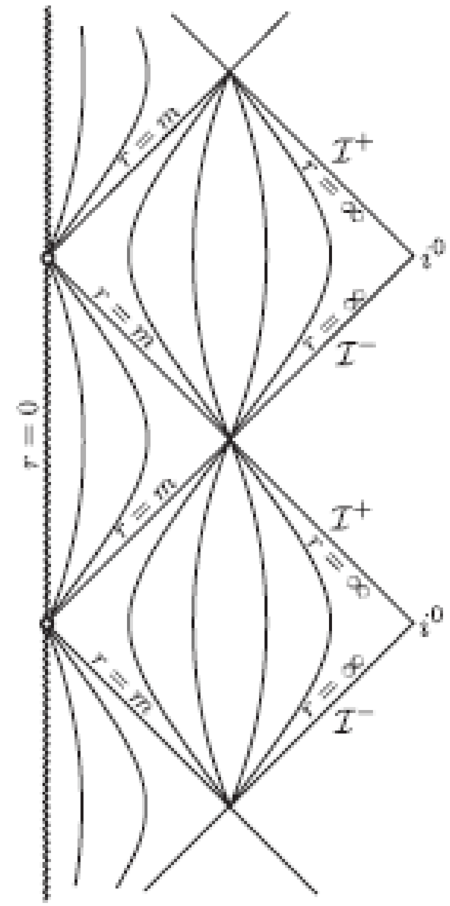
$$r_\pm = m \pm \sqrt{m^2 - (q^2 + p^2)}$$

Reissner-Nordström BHs

Conformal diagram for maxi mally extended RN BH



Conformal diagram for extre mal RN BH ($r_+ = r_-$)



Reissner-Nordström BHs in (A)dS Space

- Spherically symmetric black holes with mass M, electric charge Q and magnetic charge P

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin\theta^2\, d\phi^2)$$

$$f(r) = 1 - \frac{2m}{r} + \frac{q^2 + p^2}{r^2} \pm \frac{r^2}{L^2}$$

- Vector potential for Maxwell field

$$A_{[1]} = \frac{q}{r}dt + p(\cos\theta - \sigma)d\phi$$

- Outer (event) horizon, inner (Cauchy) horizon and cosmological horizon for dS

$$f(r) = \pm\frac{(r \pm r_c)(r - r_+)(r - r_-)(r + r_0)}{r^2}$$

Kerr-Newman-(Anti)de Sitter BHs

- Kerr-Newman (KN)-(Ant-)de Sitter ((A)dS) black holes in Boyer-Lindquist-type coordinates ((A)dS radius $L_{(A)dS} = \sqrt{3/|\Lambda|}$)

$$ds^2 = -\frac{\Delta_r}{\varrho^2}\left(dt - \frac{a\sin\theta^2}{\Xi}d\phi\right)^2 + \frac{\varrho^2}{\Delta_r}dr^2 + \frac{\varrho^2}{\Delta_\theta}d\theta^2 + \frac{\Delta_\theta \sin\theta^2}{\varrho^2}\left(adt - \frac{r^2+a^2}{\Xi}d\phi\right)^2$$

$$\varrho^2 = r^2 + a^2\cos\theta^2 \,, \qquad \Delta_\theta = 1 \mp \frac{a^2}{L^2}\cos\theta^2$$

$$\Delta_r = (r^2+a^2)\left(1 \pm \frac{a^2}{L^2}r^2\right) - 2Mr + (Q^2+P^2)\,, \qquad \Xi = 1 \mp \frac{a^2}{L^2}$$

- Mass, angular momentum, electric and magnetic charges: $\frac{M}{\Xi^2}, \frac{Ma}{\Xi^2}, \frac{Q}{\Xi}, \frac{\mathrm{P}}{\Xi}$

- Vector potential for Maxwell field

$$A_{[1]} = \frac{Qr - Pa\cos\theta}{\varrho^2}\left(dt - \frac{a\sin\theta^2}{\Xi}d\phi\right) + \frac{P(\cos\theta - \sigma)}{\Xi}d\phi$$

Schwinger Effect in Curved Spacetimes

Zaumen (‘74), Carter (‘74), Gibbons (‘75)

Damour, Ruffini (‘76)

⋮

Ruffini (‘98) dyadosphere of BH

Khriplovich (‘99)

Gabriel (‘01)

SPK, Page (‘04), (‘05), (‘08)

Ruffini, Vereshchagin, Xue (‘10)

Chen, SPK, Lin, Sun, Wu (‘12); Chen, Sun, Tang, Tsai (‘15)

Ruffini, Wu, Xue (‘13)

SPK (‘13), Cai, SPK (‘14)

SPK, Lee, Yoon (‘15); SPK (‘15)

Chen, SPK, Sun, Tang (‘16)

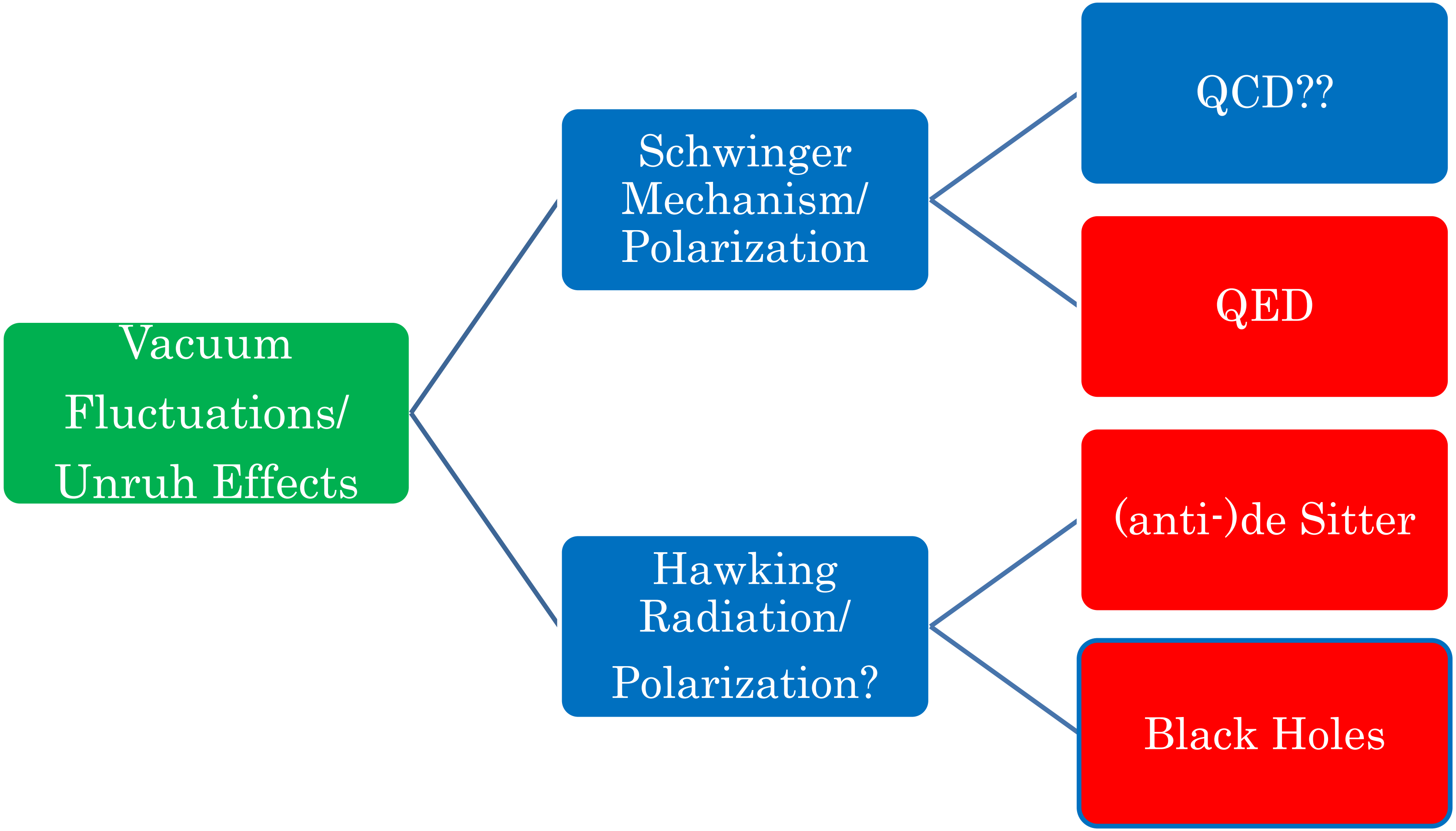
Chen, SPK (‘20), Cai, Chen, SPK, Sun (‘20), Zhang et al (‘21)

Chen, SPK (‘23), Chen et al (‘22)

Moss, Stazsak (‘23), Siahaan (‘23), ⋯

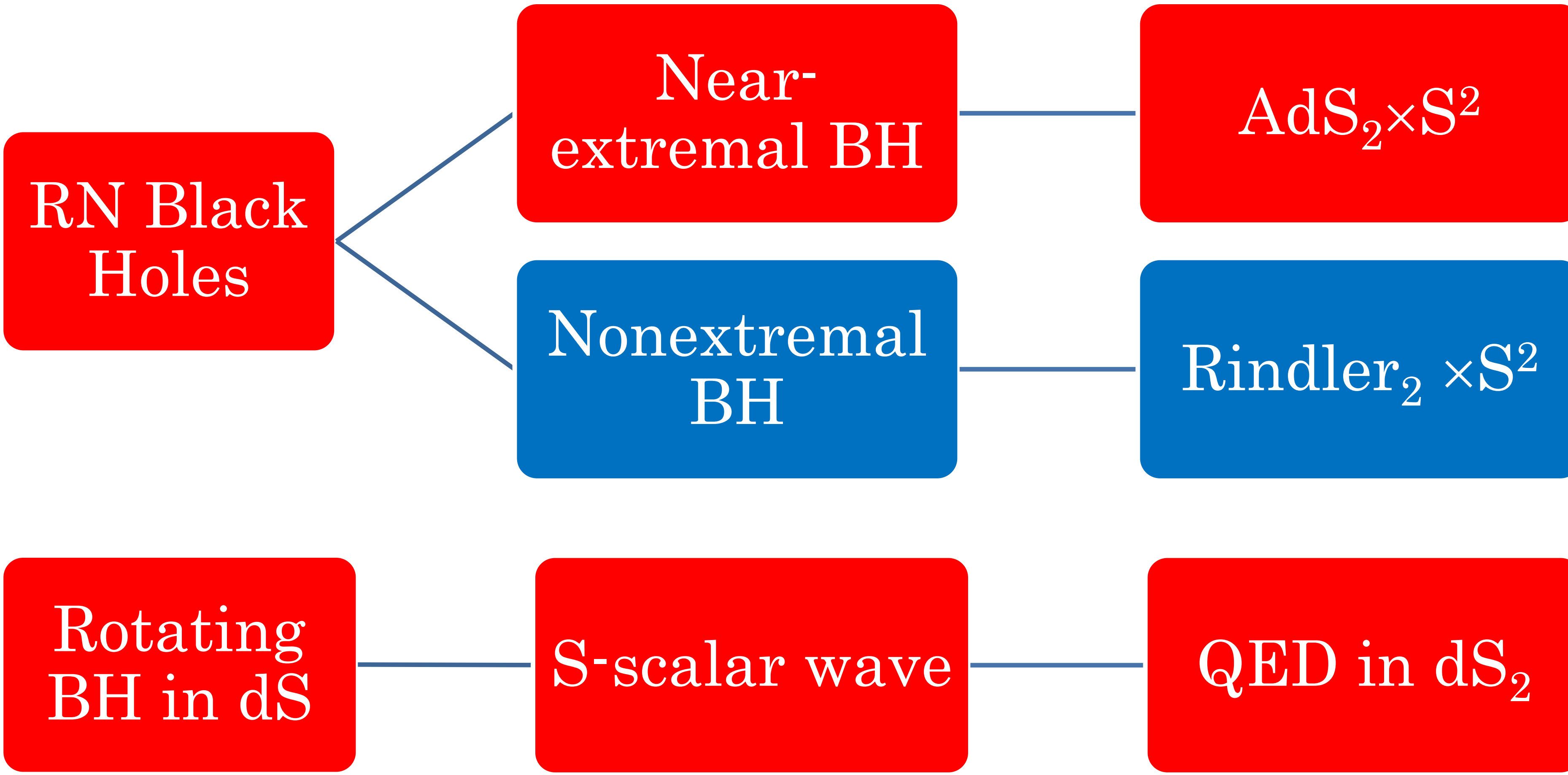
Unified Picture for Spontaneous Pair Production

Spontaneous Pair Production: Unified Picture [SPK ('07)]



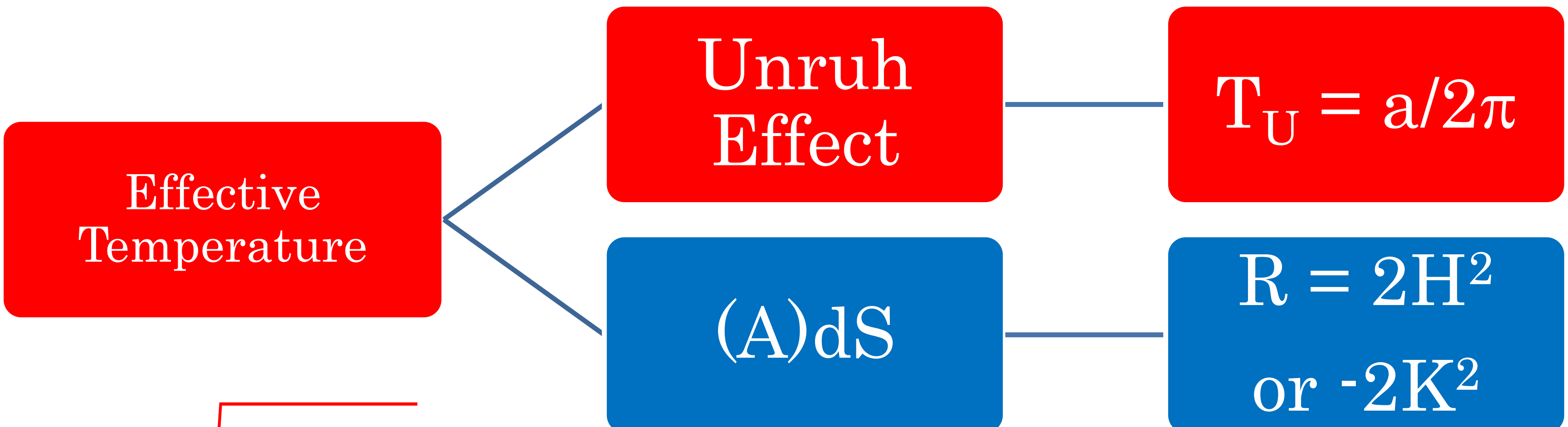
Why Schwinger Effect in (A)dS$_2$?

Near-Horizon Geometry of RN BHs



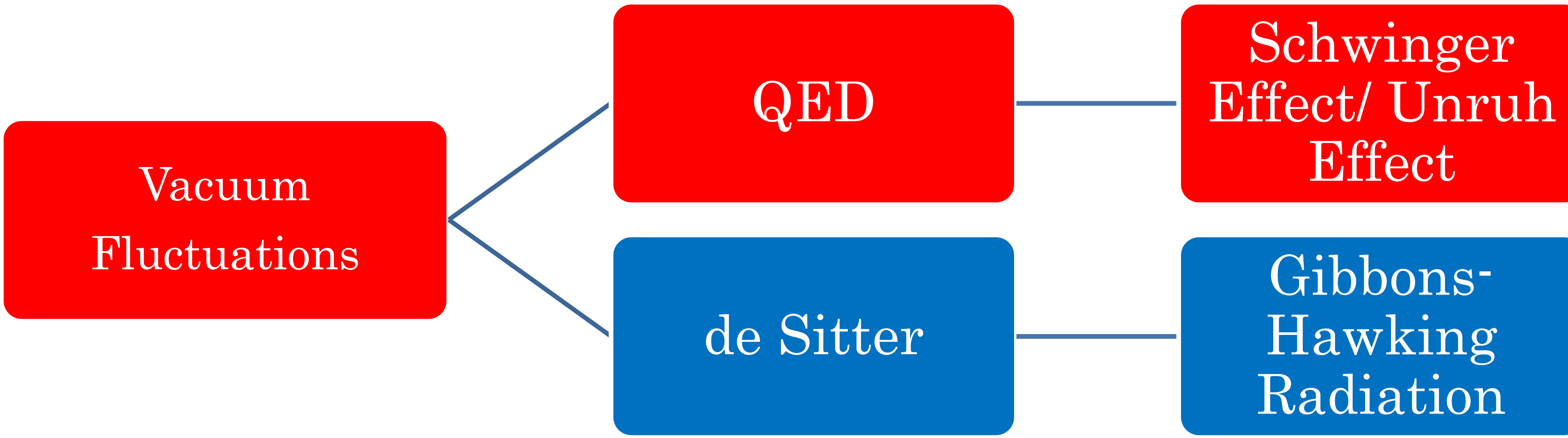
Effective Temperature for Unruh Effect in (A)dS$_2$

[Narnhofer, Peter, Thirring ('96); Deser, Levin ('97)]



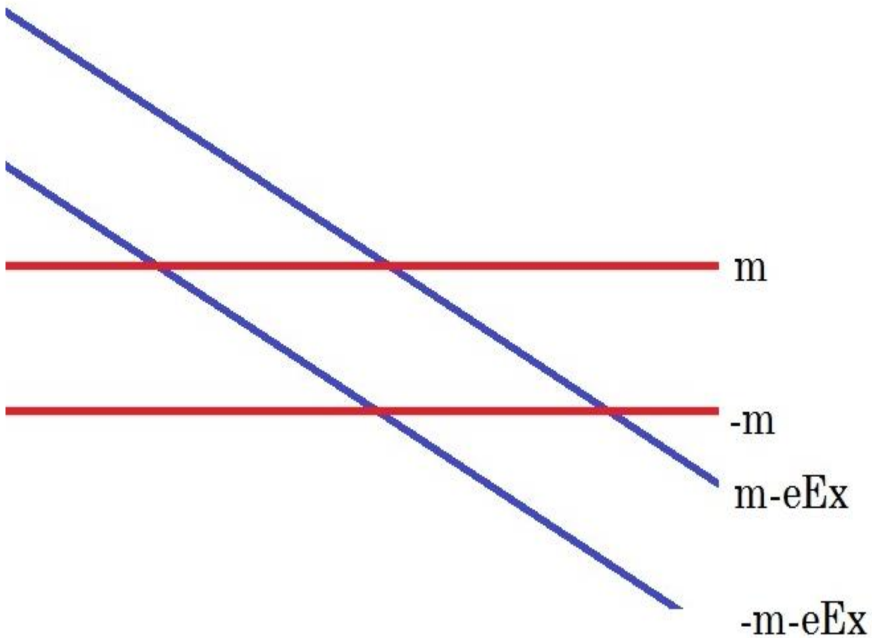
$$T_{\text{eff}} = \sqrt{T_{\text{U}}^2 + \frac{R}{8\pi^2}}$$

Schwinger Effect in (A)dS [Cai, SPK ('14)]



$$T_{\mathrm{eff}} = T_U + \sqrt{T_U^2 + T_{GH}^2}$$

What is Schwinger Effect?



- Constant E-field changes energy spectra in Minkowski spacetime:

$$\varepsilon_{\pm} = |eE|x \pm \sqrt{\vec{p}^2 + m^2}$$

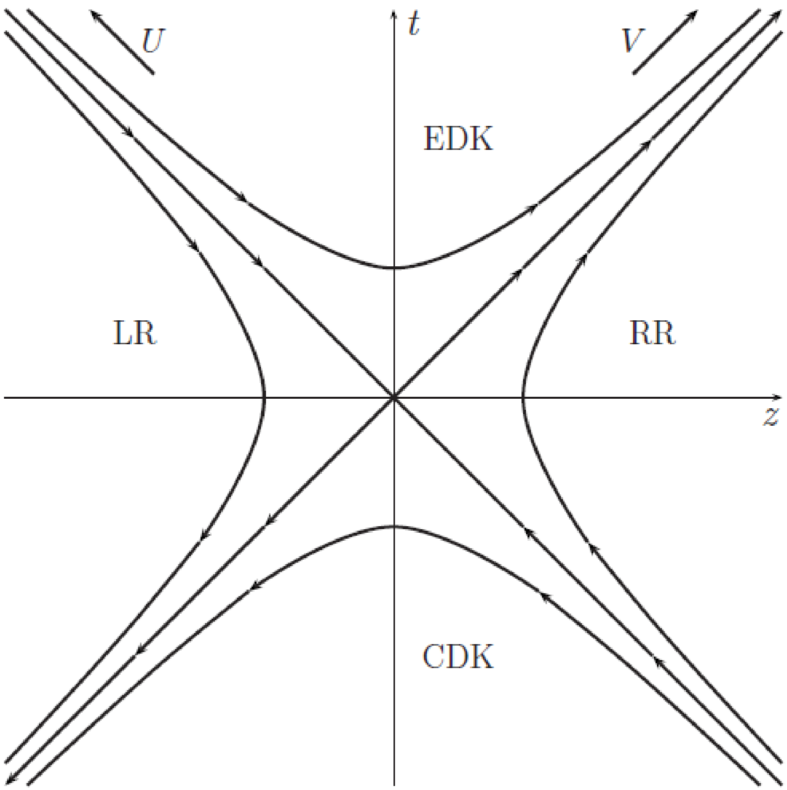
- Spontaneous creation of a particle-antiparticle pair from the Dirac sea (quantum mechanical tunneling)

$$N_S = \exp\left(-\frac{m}{2T_S}\right), \quad T_S = \frac{1}{2\pi}\left(\frac{qE}{m}\right)$$

- Critical (Schwinger) field to energetically separate the pair

$$eE_c \times \left(\frac{\hbar}{mc}\right) = mc^2$$

What is Unruh Effect?



[Crispino, Higuchi, Matsas, Rev. Mod. Phys. 80 ('08)]

- Four regions of Minkowski space:
  *right Rinder wedge (RR, an accelerating observer to the right),
  *left Rindler wedge (LR, an accelerating observer to the left),
  *expanding degenerate Kasner universe (EDK)
  *contracting degenerate Kasner universe (CDK).
- Unruh effect and temperature

$$\left\langle 0_M \left| \hat{a}_\omega^{+R} \hat{a}_\omega^{R} \right| 0_M \right\rangle = \frac{1}{e^{\omega/T_U} - 1}$$

$$T_U = \frac{[\hbar]a}{2\pi[ck_B]}$$

- Different view point: V. Belinski

Hawking Radiation & Schwinger Effect

- Hawking emission formula in charged BH [CMP ('74)]

$$N_H = \frac{\Gamma_{j\omega lm}}{e^{\frac{\omega - q_j \Phi}{T_H}} \mp 1}$$

$$T_H = \frac{1}{2\pi} \frac{\sqrt{M^2 - Q^2}}{\left(M + \sqrt{M^2 - Q^2}\right)^2}$$

- No Hawking radiation when Q = M

- Schwinger emission formula in E-field [PR ('51)]

$$N_S = \exp\left(-\frac{m}{T_S}\right)$$

$$T_S = 2 \times \frac{1}{2\pi}\left(\frac{qE}{m}\right)$$

- Heisenberg-Euler, Weisskopf, Schwinger QED actions

Connecting Schwinger Mechanism & Unruh Effect & Hawking Radiation

- An intuitive way to understand particle (pair) production

[Frolov & Novikov, *Black Hole Physics* (1998)]

$$P \cong \exp\left(-2\pi \frac{\text{energy (mass)}}{\text{force } \times \text{ Compton length}}\right) = \exp\left(-\frac{\text{energy (mass)}}{k_{\mathrm{B}}T}\right)$$

- Schwinger pair production: $F = eE, l = \frac{[\hbar]}{m[c]} \Leftrightarrow T = \frac{eE/m}{2\pi}$
- Unruh effect: $F = ma, l = \frac{[\hbar]}{m[c]} \Leftrightarrow T = \frac{a}{2\pi}$
- Hawking radiation: $F = m\kappa, l = \frac{[\hbar]}{m[c]} \Leftrightarrow T = \frac{\kappa}{2\pi}$

Complex Analysis Method for Particle Production

Mathematical Beauty behind Particle Production and Stokes Phenomena

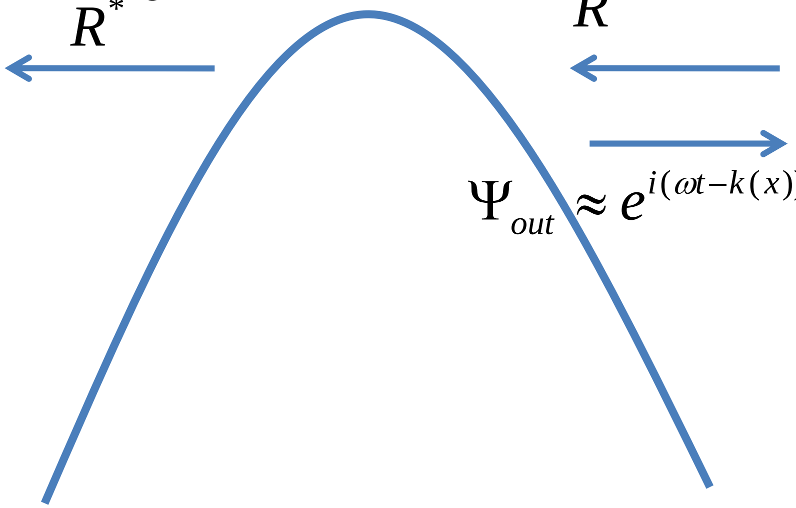
[SPK, AAPPS Bulletin 23, No. 6, 35 (2013); e-EPS, December (2013)]

At the James Scott Prize Lecture in 1939, P. A. M. Dirac emphasized the theory of functions of a complex variable as an interesting mathematical theory that fulfilled his criteria of beauty. He found this field to be of "exceptional beauty" and hence likely to lead to deep physical insight [1]. The lecture was delivered a decade after he discovered the Dirac equation, the positron was found, and the concept of the Dirac sea was well adopted.

[1] H S Kragh, DIRAC: A Scientific Biography (Cambridge University Press, 1990).

Boundary Condition for Static Systems (B-field, AdS or E-field)

$$\Psi_{out}^* \approx \frac{T^*}{R^*} e^{i(\omega t + k(x))} \qquad \Psi_{in} \approx \frac{1}{R^*} e^{i(\omega t + k(x))}$$



$$\Psi_{out} \approx e^{i(\omega t - k(x))}$$

Tunneling under the barrier

- Flux conservation

$$\left|\frac{1}{R}\right|^2 - \left|\frac{T}{R}\right|^2 = 1$$

- Bogoliubov coefficients for scalar QED

$$\alpha = \frac{1}{R^*}, \qquad \beta = \frac{T^*}{R^*}$$

- [Nikishov, NPB 21 ('70); Brezin, Itzykson, PRD 2 ('70); SPK, Page, PRD 65 ('02), 73 ('06)]

Tunneling Interpretation

- The motion of charged particle in a constant electric field in the Coulomb gauge is the tunneling problem under the potential barrier.
- The instanton action method [SPK, Page, PRD 65 ('02), 73 ('06)]: pair production rate is given by the tunneling probability under the barrier

$$[-\partial^2/\partial x^2 - q^2(x)]\phi_\omega(x) = 0$$

$$q^2(x) = (\omega + qEx)^2 - m^2 - 2i\sigma qE = (qE)^2(x - x_+)(x - x_-)$$

$$P = e^{-S}; S = 2\int_{x_-}^{x_+} q(x)dx = \pi \frac{m^2 + k_\perp^2}{qE}$$

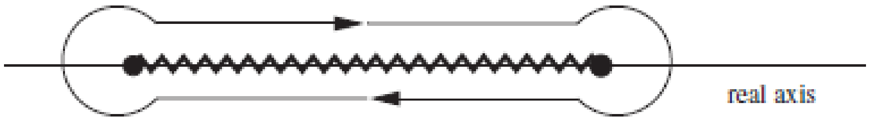
Tunneling Interpretation

- Complex analysis through a contour integral [SPK, Page, PRD 75 ('07)]

$$[-\partial^2/\partial z^2 - q^2(z)]\phi_\omega(z) = 0$$

$$q(z) = qE\sqrt{(z-z_+)(z-z_-)} = qE\left[z - \frac{z_+ + z_-}{2} - \frac{(z_+ - z_-)^2}{8z} + \cdots\right]$$

$$S = -i\oint_C \sqrt{q(z)}dz = \pi\frac{m^2 + k_\perp^2}{qE}$$



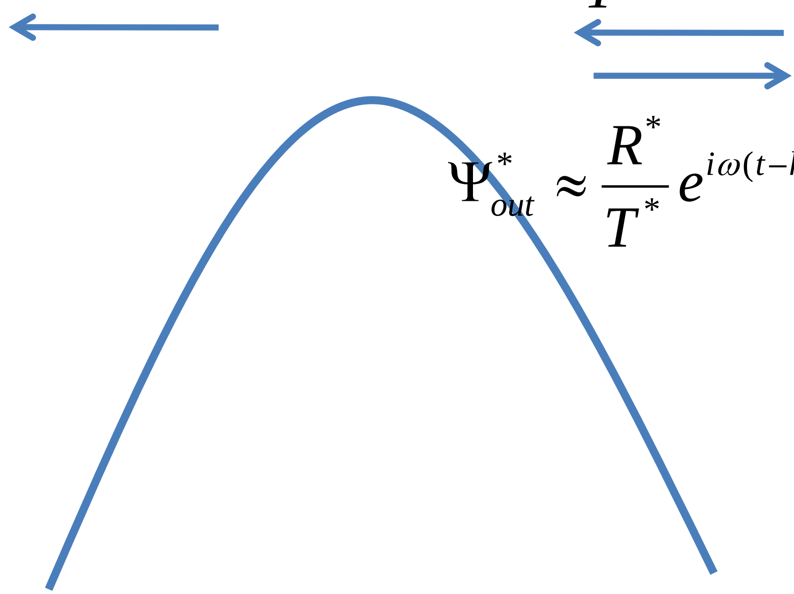
- Higher correction terms [Froman, Froman, NPA147 ('70)] yield zero residues and thus lead to the exact formula.

Boundary Condition for Nonstatic Systems (E-field, dS or B-field)

$$\Psi_{in} \approx e^{i(\omega(t)+kx)}$$

$$\Psi_{out} \approx \frac{1}{T^*} e^{i(\omega(t)+kx)}$$

$$\Psi^*_{out} \approx \frac{R^*}{T^*} e^{i\omega(t-kx)}$$



Scattering over the barrier

- Flux conservation

$$\left|\frac{1}{T}\right|^2 - \left|\frac{R}{T}\right|^2 = 1$$

- Bogoliubov coefficients for scalar QED

$$\alpha = \frac{1}{T^*}, \qquad \beta = \frac{R^*}{T^*}$$

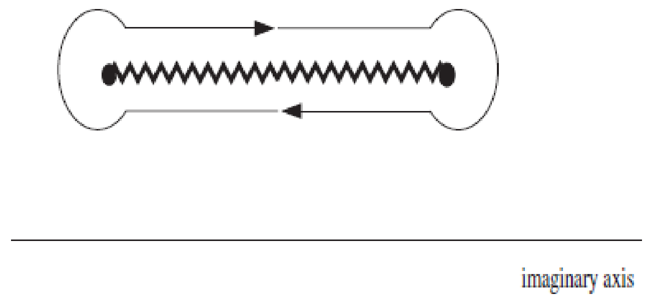
Scattering Interpretation

- Vector gauge $A_\mu = (0,0,0,-\mathrm{Et})$ and contour integral [SPK, Page, PRD 75 ('07)]

$$[\partial^2/\partial z^2 + \omega^2(z)]\phi_{\omega k}(z) = 0$$

$$\omega(z) = qE\sqrt{(z-z_+)(z-z_-)} = qE\left[z - \frac{z_+ + z_-}{2} - \frac{(z_+ - z_-)^2}{8z} + \cdots\right]$$

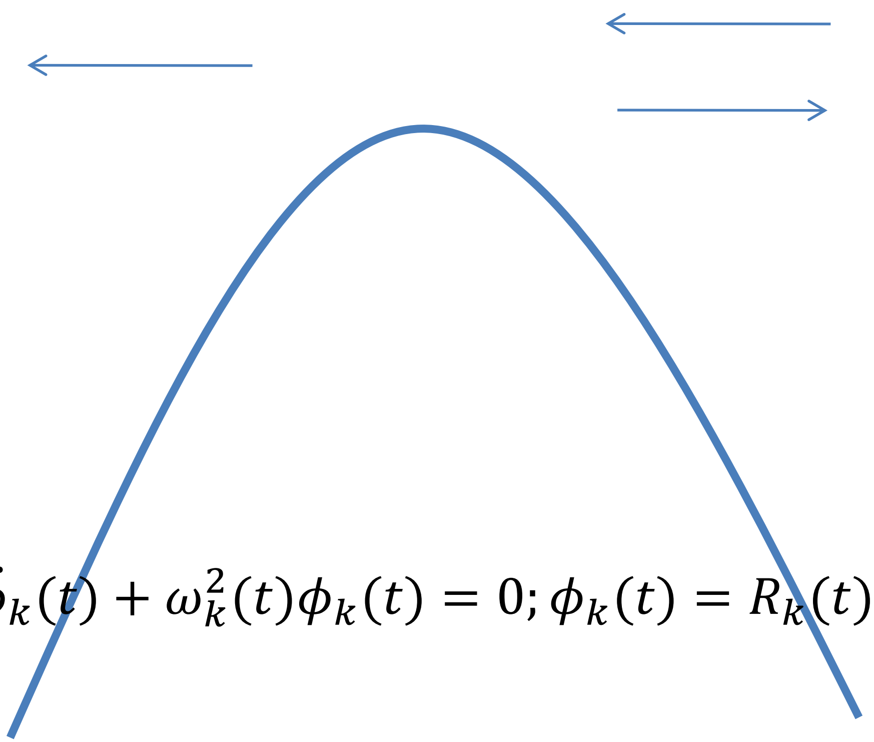
$$S = i\oint_C \omega(z)dz = \pi\frac{m^2 + k_\perp^2}{qE}$$

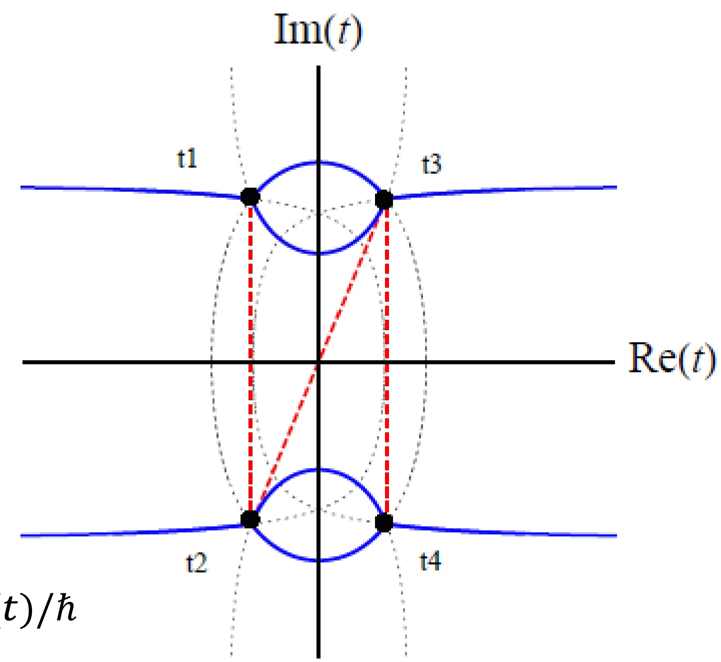


- Higher correction terms [Froman, Froman, NPA147 ('70)] yield zero residues and thus lead to the exact formula.

Scattering Picture & Stokes Phenomenon

Scattering over the Barrier **Stokes Phenomenon**





$$\ddot{\phi}_k(t) + \omega_k^2(t)\phi_k(t) = 0; \phi_k(t) = R_k(t)e^{iS_k(t)/\hbar}$$

[Fig. from Dumlu, Dunne, PRL 104 ('10); anti-Stokes lines (solid blue lines) and Stokes lines (dotted red lines)]

Boson & Fermion Production

- In the phase-integral method, the mean number of pairs with one pair of turning points of complex action [SPK, Page, PRD 75 (‘07); SPK, JHEP 09 (‘10)]: $N = e^{-2\,\mathrm{Im}\,S}$
- For gauge field with two pairs of turning points, the mean number of boson pairs [Dumlu & Dunne, PRL 104 (‘10)]
$$N_{\text{boson}} = e^{-2\,\mathrm{Im}\,S(I)} + e^{-2\,\mathrm{Im}\,S(II)} - 2\cos(\,\mathrm{Re}\,S\,(I, II))e^{-\,\mathrm{Im}\,S(I) - \mathrm{Im}\,S(II)}$$
$$\approx 4\sin^2(\,\mathrm{Re}\,S)e^{-2\,\mathrm{Im}\,S}$$
- The mean number of fermion pairs
$$N_{\text{fermion}} = e^{-2\,\mathrm{Im}\,S(I)} + e^{-2\,\mathrm{Im}\,S(II)} + 2\cos(\,\mathrm{Re}\,S\,(I, II))e^{-\,\mathrm{Im}\,S(I) - \mathrm{Im}\,S(II)}$$
$$\approx 4\cos^2(\,\mathrm{Re}\,S)e^{-2\,\mathrm{Im}\,S}$$

Phase-Integral for Particle Production

- The particle production rate in the in-in formalism [SPK, PLB 725 ('13); JPS Conf. Proc. 1 ('14)]

$$\mathcal{N}_\alpha = \left| \sum_J \left\langle 0_\alpha, t_0 \middle| 0_\alpha, C_J^{(1)} \right\rangle^2 \right| = \left| \sum_J \exp\left(-i \oint_{C_J^{(1)}(t_0)} \omega_\alpha(z) dz \right) \right|$$

- A charged particle in a constant electric field or a massive particle in a de Sitter space:

$$\omega(z) = f(z)\sqrt{(z - z_0)(z - z_0^*)} \qquad (f(z) = \text{analytic function})$$

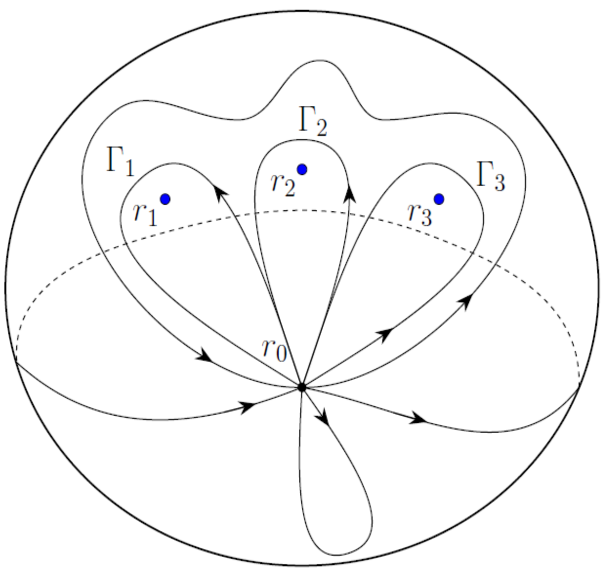
- Schwinger pair production and de Sitter radiation result from the residue theorem and Stokes phenomenon in the global coordinates [SPK, PRD 88 ('13)]

$$N_{\vec{k}} = \exp[-2\pi \mathrm{Res}[\omega(z)]] = \begin{cases} \exp\left[-\pi \dfrac{m^2 + k_\perp^2}{qE}\right] \\ \exp\left[-2\pi \dfrac{m}{H}\right] \end{cases}$$

Monodromy Method

[Chen, Ishige, SPK, Takayasu, Wei, 2210.14792]

Poles on Riemann sphere Mean number



- Riemann equation

$$y''(r) + \left(\frac{1-\rho_1-\rho_2}{r-r_1} + \frac{1-\sigma_1-\sigma_2}{r-r_2}\right) y'(r) + \frac{1}{(r-r_1)(r-r_2)}\left(\frac{\rho_1\rho_2(r_1-r_2)}{r-r_1} + \frac{\sigma_1\sigma_2(r_2-r_1)}{r-r_2} + \tau_1\tau_2\right) y(r) = 0$$

- Riemann P-function

$$P\begin{pmatrix} z_1 & z_2 & \infty \\ \rho_1 & \sigma_1 & \tau_1\,; r \\ \rho_2 & \sigma_2 & \tau_2 \end{pmatrix}, \qquad \sum_i \rho_i + \sigma_i + \tau_i = 1$$

- Mean number

$$N = \frac{\cos\pi(\sigma_1-\sigma_2+\tau_1-\tau_2) - \cos\pi(\sigma_1-\sigma_2-\tau_1+\tau_2)}{\cos\pi(\sigma_1-\sigma_2-\tau_1+\tau_2) + \cos\pi(\rho_1-\rho_2)}$$

Monodromy for Near-extremal RN BHs

- Charged scalar in near-horizon of near-extremal RN BHs

$$\frac{d^2R}{d\rho^2}+\frac{2\rho}{\rho^2-B^2}\frac{dR}{d\rho}+\left(\frac{Q^2(\omega Q-q\rho)^2}{(\rho^2-B^2)^2}-\frac{m^2Q^2}{\rho^2-B^2}\right)R=0$$

- Riemann P-function

$$P\begin{pmatrix} -B & B & \infty & \\ i\alpha_- & i\alpha_+ & \frac{1}{2}+i\alpha_\infty & ;\rho \\ -i\alpha_- & -i\alpha_+ & \frac{1}{2}-i\alpha_\infty & \end{pmatrix}$$

$$\alpha_\pm=Q^2\frac{\omega}{B}\mp qQ\ ,\alpha_\infty=\sqrt{(q^2-m^2)Q^2-\lambda_l-\frac{1}{4}}$$

- Mean number for Schwinger pair production

$$N=\frac{\sinh(2\pi\alpha_+)\sinh(2\pi\alpha_\infty)}{\cosh\pi(\alpha_+ +\alpha_- -\alpha_\infty)\cosh\pi(\alpha_+ -\alpha_- -\alpha_\infty)}$$

QED in (A)dS$_2$

Schwinger formula in (A)dS_2

- (A)dS metric and the gauge potential for E

$$ds^2 = -dt^2 + e^{2Ht}dx^2, A_1 = -(E/H)(e^{Ht} - 1)$$
$$ds^2 = -e^{2Kx}dt^2 + dx^2, A_0 = -(E/K)(e^{Kx} - 1)$$

- Schwinger formula (mean number) for scalars in dS_2 [Garriga ('94); SPK, Page ('08)] and in AdS_2 [Pioline, Troost ('05); SPK, Page ('08)]

$$N = e^{-S}, \qquad S = \frac{\pi m^2}{qE}\left(\frac{2 - \frac{R}{4m^2}}{1 + \sqrt{1 + \frac{m^2 R}{2(qE)^2} - \frac{R^2}{16(qE)^2}}}\right)$$

Effective Temperatures

- Effective temperature for accelerating observer in (A)dS$_2$
[Narnhofer, Peter, Thirring ('96); Deser, Levin ('97)]

$$N = e^{-m/T_{\text{eff}}}, \qquad T_{\text{eff}} = \sqrt{T_{\text{U}}^2 + \frac{R}{8\pi^2}}, \qquad R = 2H^2, (-2K^2)$$

- Effective temperature for Schwinger formula in (A)dS$_2$
[Cai, SPK ('14)]

$$N = e^{-\bar{m}/T_{\text{eff}}}, \qquad \bar{m} = \sqrt{m^2 - \frac{R}{8}}, \qquad T_{\text{U}} = \frac{qE/\bar{m}}{2\pi}, \qquad T_{\text{GH}} = \frac{H}{2\pi}$$

$$T_{\text{dS}} = \sqrt{T_{\text{U}}^2 + T_{\text{GH}}^2} + T_{\text{U}}, \qquad T_{\text{AdS}} = \sqrt{T_{\text{U}}^2 + \frac{R}{8\pi^2}} + T_{\text{U}}$$

Scalar QED Action in dS_2

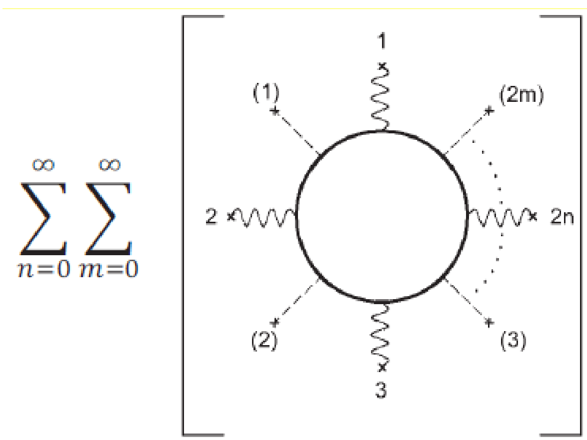
- Mean number for pair production and vacuum polarization from the in-out formalism [Cai, SPK ('14)]

$$N_{dS} = \frac{e^{-(S_\mu - S_\lambda)} + e^{-2S_\mu}}{1 - e^{-2S_\mu}}, \qquad 2\,\mathrm{Im}\,W_{dS}^{(1)} = \ln(1 + N_{dS})$$

$$L_{dS}^{(1)} = \frac{H^2 S_\mu}{2(2\pi)} P \int_0^\infty \frac{ds}{s} \left[e^{-(S_\mu - S_\lambda)s/2\pi} \left(\frac{1}{\sin(s/2)} - \overbrace{\left(\frac{2}{s}\right)}^{\text{Schwinger subtraction}} \right) - e^{-S_\mu s/\pi} \left(\frac{\cos(s/2)}{\sin(s/2)} - \frac{2}{s} \right) \right]$$

$$S_\mu = 2\pi \sqrt{\left(\frac{qE}{H^2}\right)^2 + \left(\frac{m}{H}\right)^2 - \frac{1}{4}}, \qquad S_\lambda = 2\pi \frac{qE}{H^2}$$

Large order perturbation



Spinor QED Action in dS_2

- Mean number for pairs and vacuum polarization [SPK ('15)]

$$N_{dS}^{sp} = \frac{e^{-(S_\mu - S_\lambda)} - e^{-2S_\mu}}{1 - e^{-2S_\mu}}, \qquad 2\,\mathrm{Im}\,W_{dS}^{(1)} = -\ln\left(1 - N_{dS}^{sp}\right)$$

$$N_{dS}^{sp} = -\frac{H^2 S_\mu}{2\pi} P \int_0^\infty \frac{ds}{s}\left(e^{-(S_\mu - S_\lambda)s/2\pi} - e^{-S_\mu s/\pi}\right)\left(\cot\left(\frac{s}{2}\right) - \frac{2}{s}\right)$$

$$S_\mu = 2\pi\sqrt{\left(\frac{qE}{H^2}\right)^2 + \left(\frac{m}{H}\right)^2}, \qquad S_\lambda = 2\pi\frac{qE}{H^2}$$

Scalar QED Action in AdS_2

- Mean number for pair production, violation of BF bound & vacuum polarization [Cai, SPK ('14)]

$$N_{\mathrm{AdS}}^{\mathrm{sc}} = \frac{e^{-(S_\kappa - S_\nu)} - e^{-(S_\kappa + S_\nu)}}{1 + e^{-(S_\kappa + S_\nu)}}, 2\,\mathrm{Im}W_{\mathrm{AdS}}^{(1)} = \ln\left(1 + N_{\mathrm{AdS}}\right)$$

$$L_{\mathrm{AdS}}^{\mathrm{sc}} = -\frac{K^2 S_\nu}{2(2\pi)} P\int_0^\infty \frac{ds}{s} e^{-S_\kappa s/2\pi} \cosh(S_\nu s/2\pi)\left(\frac{1}{\sin(s/2)} - \frac{2}{s} - \frac{s}{12}\right)$$

$$S_\nu = 2\pi\sqrt{\left(\frac{qE}{K^2}\right)^2 - \left(\frac{m}{K}\right)^2 - \frac{1}{4}}, \qquad S_\kappa = 2\pi\frac{qE}{K^2}$$

$\left({}^{qE}/_{K}\right)^2 \geq m^2 + {}^{K^2}/_{4}$ (violation of BF bound)

Spinor QED Action in AdS_2

- Mean number for pairs, violation of BF bound & vacuum polarization [SPK ('15)]

$$N^{\rm sp}_{\rm AdS} = \frac{e^{-(S_\kappa - S_\nu)} - e^{-(S_\kappa + S_\nu)}}{1 - e^{-(S_\kappa + S_\nu)}}, 2\,{\rm Im} W^{\rm sp}_{\rm AdS} = -\ln\left(1 - N^{\rm sp}_{\rm AdS}\right)$$

$$L^{\rm sp}_{\rm AdS} = -\frac{K^2 S_\nu}{2\pi} P\int_0^\infty \frac{ds}{s}\left(e^{-(S_\kappa - S_\nu)s/2\pi} - e^{-(S_\kappa + S_\nu)s/2\pi}\right)\left(\cot(\frac{s}{2}) - \frac{2}{s} + \frac{s}{6}\right)$$

$$S_\nu = 2\pi\sqrt{\left(\frac{qE}{K^2}\right)^2 - \left(\frac{m}{K}\right)^2}, \qquad S_\kappa = 2\pi\frac{qE}{K^2}$$

$$\left({}^{qE}/_{K}\right)^2 \geq m^2$$

Schwinger Effect in Near-extremal Black Holes

Schwinger Effect in RN BHs

- Thermal interpretation of Schwinger formula for charged scalars and fermions harmonics [Chen, SPK, Lin, Sun, Wu ('12); Chen, Sun, Tang, Tsai ('15); SPK, Lee, Yoon ('15); SPK ('15)]

$$N_{NBH} = \underbrace{\left(\frac{e^{-\frac{\bar{m}}{T_{RN}}} - e^{-\frac{\bar{m}}{\bar{T}_{RN}}}}{1 \pm e^{-\frac{\bar{m}}{\bar{T}_{RN}}}} \right)}_{\text{Schwinger effect in AdS}_2} \times e^{\frac{\bar{m}}{T_{RN}}} \times \underbrace{\left(\frac{e^{-\frac{\bar{m}}{T_{RN}}} (1 \mp e^{-\frac{\omega - qA_0}{T_H}})}{1 + e^{-\frac{\omega - qA_0}{T_H}} e^{-\frac{\bar{m}}{T_{RN}}}} \right)}_{\text{Schwinger effect in Rindler space}}$$

Schwinger effect in AdS_2
Cai & SPK ('14)

Schwinger effect in Rindler space
Gabriel & Spindel ('00)
Hawking radiation of charges

$$T_{RN} = T_U + \sqrt{T_U^2 - (\frac{1}{2\pi Q})^2}, \qquad \bar{T}_{RN} = T_U - \sqrt{T_U^2 - (\frac{1}{2\pi Q})^2}$$

$$T_U = \frac{qE_H/\bar{m}}{2\pi} = \frac{q}{2\pi \bar{m} Q}$$

Schwinger Effect in Kerr NHs

- Thermal interpretation of Schwinger formula for charged scalars (spinors) in spheroidal harmonics in near-extremal KN BH [Chen, SPK, Sun, Tang ('16)]

$$N_{NKN} = \underbrace{\left(\frac{e^{-\frac{\bar{m}}{T_{KN}}} - e^{-\frac{\bar{m}}{\bar{T}_{KN}}}}{1 \pm e^{-\frac{\bar{m}}{\bar{T}_{KN}}}} \right)}_{\text{Schwinger effect in AdS}_2} \times e^{\frac{\bar{m}}{T_{KN}}} \times \underbrace{\left(\frac{e^{-\frac{\bar{m}}{T_{KN}}} \left(1 \mp e^{-\frac{\omega_t - q\Phi_H - n\Omega_H}{T_H}}\right)}{1 + e^{-\frac{\omega_t - q\Phi_H - n\Omega_H}{T_H}} e^{-\frac{\bar{m}}{T_{KN}}}} \right)}_{\text{Schwinger effect in Rindler space}}$$

$$T_{KN} = T_U + \sqrt{T_U^2 + \frac{R_2}{8\pi^2}}, \qquad \bar{T}_{KN} = T_U - \sqrt{T_U^2 + \frac{R_2}{8\pi^2}}$$

$$T_U = \frac{qQ^3 - 2nar_0}{2\pi\bar{m}(r_0^2 + a^2)^2}, \qquad R_2 = -\frac{2}{r_0^2 + a^2}, \qquad \bar{m} = \sqrt{m^2 - \frac{\lambda + 1/4}{2} R_2}$$

Schwinger Effect in Dyon RN BH in (A)dS

- Thermal interpretation of Schwinger formula for charged scalars in spheroidal harmonics in near-extremal KN BHs [Chen, SPK, ('20), Cai, Chen, SPK, ('20)]

$$N_{NS} = \underbrace{\left(\frac{e^{-\frac{\bar{m}}{T_S}} - e^{-\frac{\bar{m}}{\tilde{T}_S}}}{1 \pm e^{-\frac{\bar{m}}{\tilde{T}_S}}} \right)}_{\text{Schwinger effect in AdS2}} \times e^{\frac{\bar{m}}{T_S}} \times \underbrace{e^{-\frac{\bar{m}}{S}} \left(\frac{1 \mp e^{-\frac{\omega_t + q\Phi_H + p\bar{\Phi}_H}{T_H}}}{1 + e^{-\frac{\omega_t + q\Phi_H + p\bar{\Phi}_H}{T_H}} e^{-\frac{\bar{m}}{T_S}}} \right)}_{\text{Schwinger effect with chemical potentials in Rindler space}}$$

$$T_S = T_U + \sqrt{T_U^2 + \frac{R}{8\pi^2}}, \qquad \tilde{T}_S = T_U - \sqrt{T_U^2 - \frac{R}{8\pi^2}}$$

$$T_U = \frac{qQ + pP}{2\pi \bar{m} R_S^2}, \qquad R = -\frac{2}{R_S^2}, \qquad \bar{m} = \sqrt{1 + \left(\frac{l + 1/2}{m R_S} \right)^2}$$

Dyon Production in KN-(A)dS BHs

- Thermal interpretation of Schwinger formula for charged scalars in spheroidal harmonics in near-extremal KN BH [Chen, SPK, (‘23)]

$$N_{NKN} = \underbrace{\left(\frac{e^{-\frac{\bar{m}}{T_{KN}}} - e^{-\frac{\bar{m}}{\tilde{T}_{KN}}}}{1 \pm e^{-\frac{\bar{m}}{\tilde{T}_{KN}}}} \right)}_{\text{Schwinger effect in AdS2}} \times e^{\frac{\bar{m}}{T_{KN}}}$$

$$\times \underbrace{e^{-\frac{\bar{m}}{T_{KN}}} \left(\frac{1 - e^{-\frac{\omega_t + q\Phi_H + p\bar{\Phi}_H + n\Omega_H}{T_H}}}{1 + e^{-\frac{\omega_t + q\Phi_H + p\bar{\Phi}_H + n\Omega_H}{T_H}} e^{-\frac{\bar{m}}{T_{KN}}}} \right)}_{\text{Schwinger effect with chemical potentials in Rindler space}}$$

$$T_{KN} = T_U + \sqrt{T_U^2 + \frac{\Delta_0}{4\pi^2 (r_0^2 + a^2)}}, \qquad \tilde{T}_{KN} = T_U - \sqrt{T_U^2 + \frac{\Delta_0}{4\pi^2 (r_0^2 + a^2)}}$$

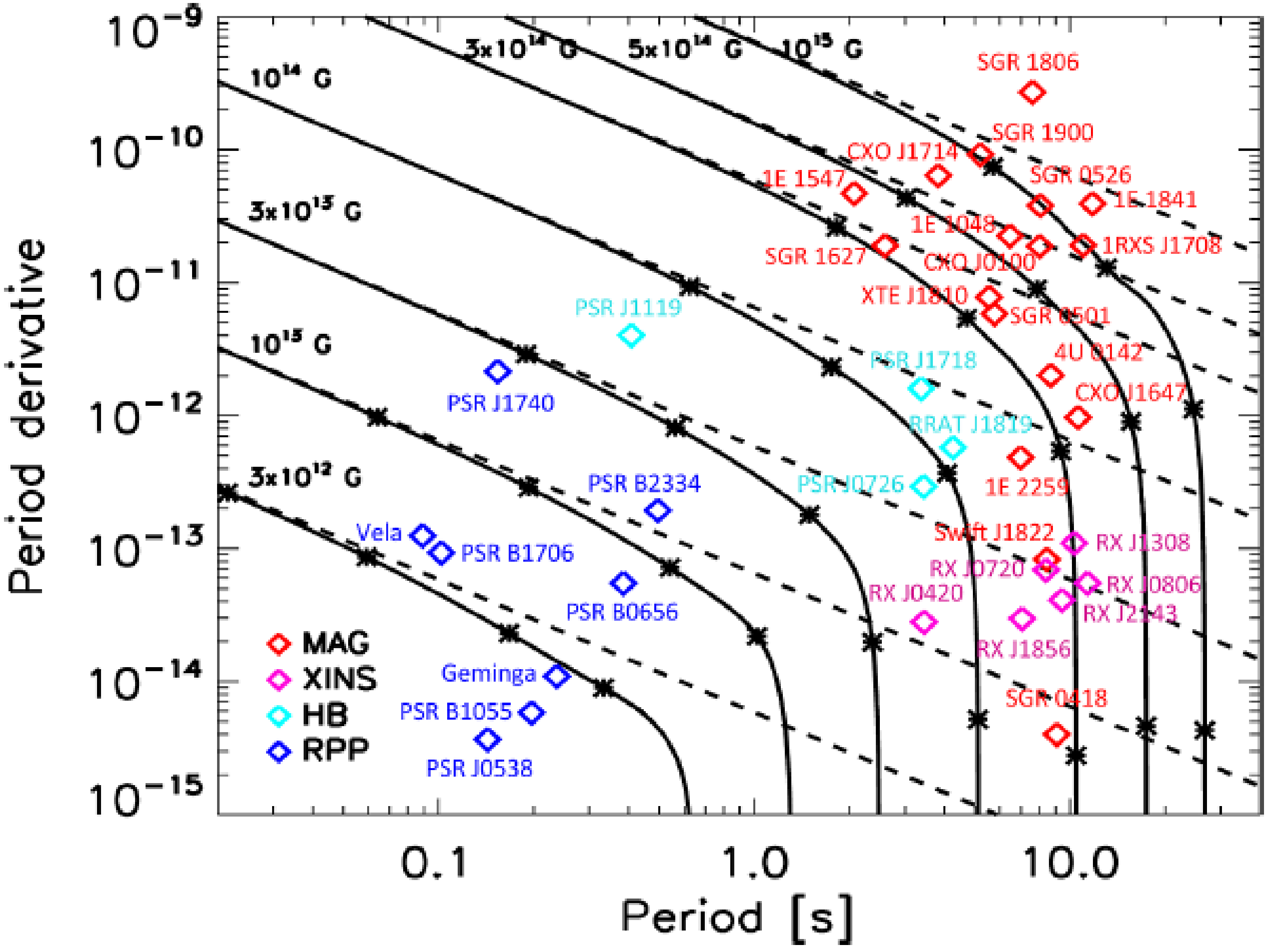
$$T_U = \frac{(qQ + pP)(r_0^2 - a^2) - 2nar_0}{2\pi\bar{m}(r_0^2 + a^2)^2}, \qquad \bar{m} = \sqrt{1 + \frac{\lambda + \Delta_0/4}{m^2 (r_0^2 + a^2)}}$$

Nonlinear Electrodynamics in Astrophysics

When does nonlinear electrodynamics matter in physics?

- Nonlinear electrodynamics means electromagnetic theory beyond the Maxwell theory.
- When the magnetic or electric length $\left({}^{1}\!/_{\sqrt{qB}},\ {}^{1}\!/_{\sqrt{qE}}\right)$ is comparable to or shorter than the characteristic length scale of physical systems, the electromagnetic theory becomes nonlinear.

Magnetars



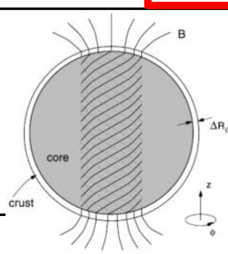
Evolutionary tracks in $P - \dot{P}$ diagram with model for mass and radius, with initial magnetic fields; asterisks mark the real ages $10^3, 10^4, 10^5, 5 \times 10^5 yr$; dashed lines without magnetic field decay [Vignano et al, MNRAS 343 ('13)]

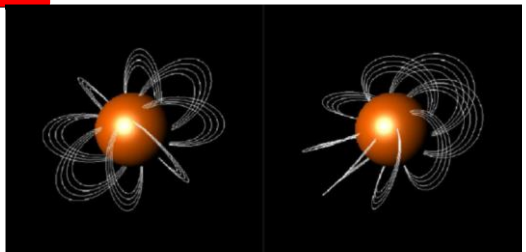
The McGill Magnetar Catalog

[Olausen, Kaspi, APJSS 212 ('14)]

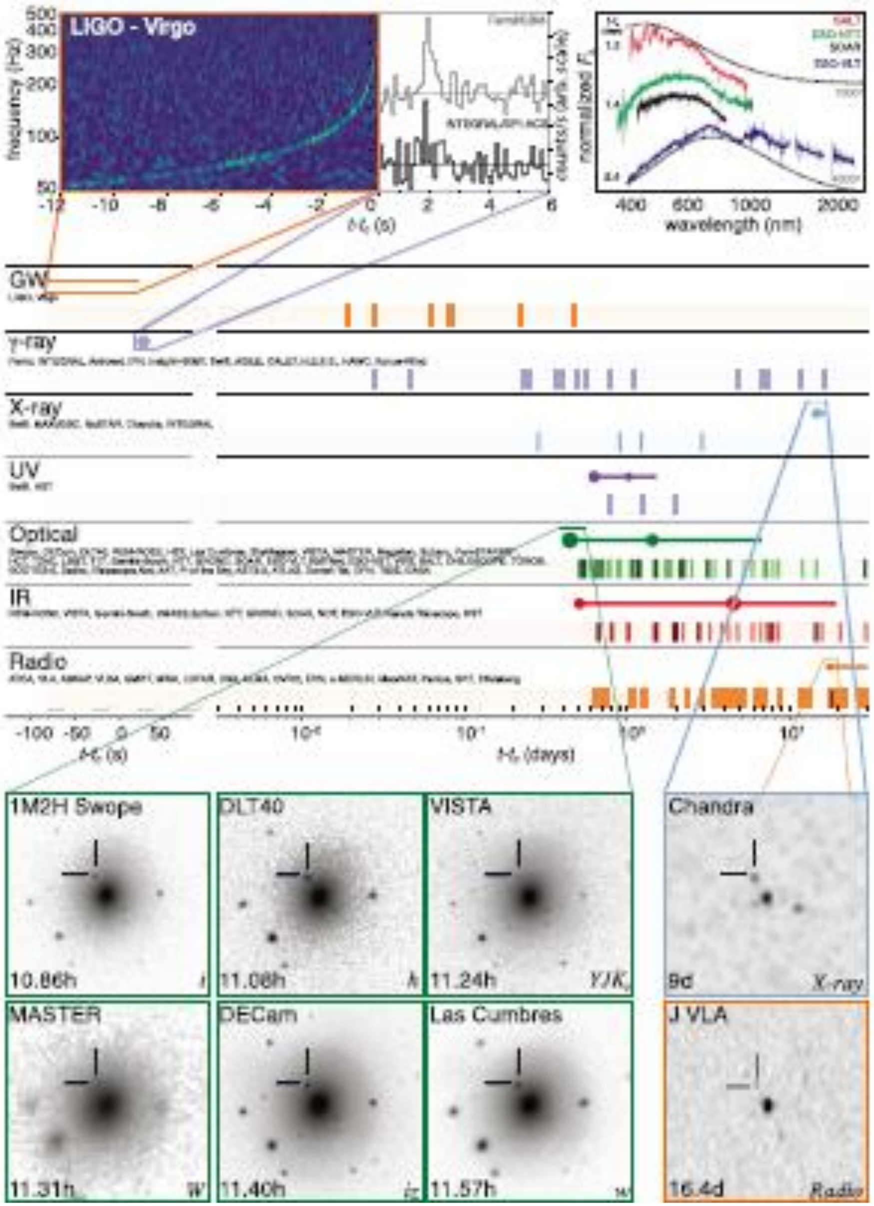
Table 2
Magnetar Timing Properties

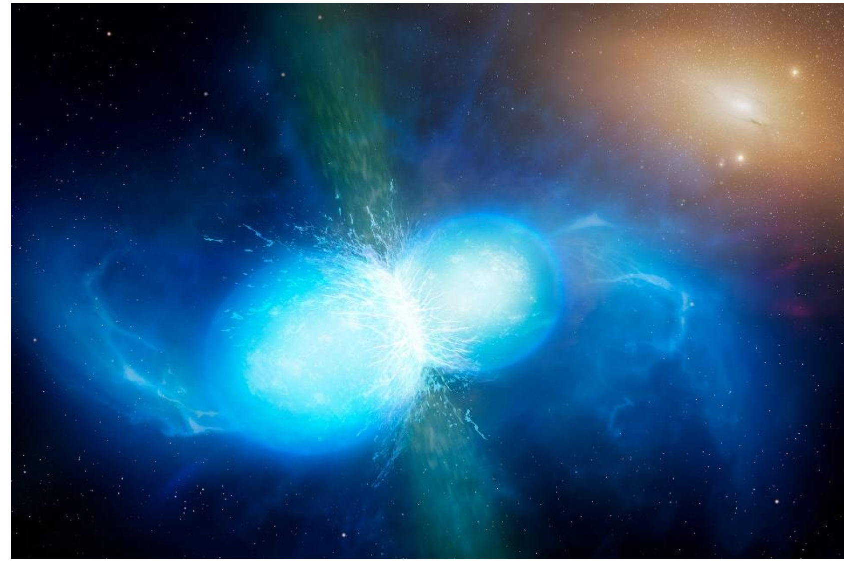
Name	P (s)	Epoch (MJD)	$\dot{P}$ (10^{-11} s s^{-1})	$\dot{P}$ Range (MJD)	Method[a]	B (10^{14} G)	$\dot{E}$ (10^{33} erg s^{-1})	τ_c (kyr)	References
CXOU J010043.1−721134	8.020392(9)	53032	1.88(8)	52044–53033	A	3.9	1.4	6.8	1
4U 0142+61	8.68832877(2)	51704	0.20332(7)	51610–53787	ED	1.3	0.12	68	2
SGR 0418+5729	9.07838822(5)	54993	0.0004(1)	54993–56164	E	0.061	0.00021	36000	3
SGR 0501+4516	5.76209653(3)	54750	0.582(3)	54700–54940	ED	1.9	1.2	16	4
SGR 0526−66	8.0544(2)	54414	3.8(1)	52152–54414	A	5.6	2.9	3.4	5
1E 1048.1−5937	6.4578754(25)	54185.9	~2.25	50473–54474	A	3.9	3.3	4.5	6
1E 1547.0−5408	2.0721255(1)	54854	~4.77	54743–55191	A	3.2	210	0.69	7
PSR J1622−4950	4.3261(1)	55080	1.7(1)	54939–55214	A	2.7	8.3	4.0	8
SGR 1627−41	2.594578(6)	54734	1.9(4)	54620–54736	A	2.2	43	2.2	9, 10
CXOU J164710.2−455216	10.610644(17)	53999.1	<0.04	53513–55857	A	<0.66	<0.013	>420	11
1RXS J170849.0−400910	11.003027(1)	53635.7	1.91(4)	53638–54015	ED	4.6	0.57	9.1	12
CXOU J171405.7−381031	3.825352(4)	55272	6.40(5)	54856–55272	A	5.0	45	0.95	13
SGR J1745−2900	3.7635537(2)	56424.6	0.661(4)	56406–56480	E	1.6	4.9	9.0	14
SGR 1806−20	7.547728(17)	53097.5	~49.5	52021–53098	A	20	45	0.24	15
XTE J1810−197	5.5403537(2)	54000	0.777(3)	53850–54127	E	2.1	1.8	11	16
Swift J1822.3−1606	8.43771958(6)	55761	0.0306(21)	55758–55991	ED	0.51	0.020	440	17
SGR 1833−0832	7.5654084(4)	55274	0.35(3)	55274–55499	ED	1.6	0.32	34	18
Swift J1834.9−0846	2.4823018(1)	55783	0.796(12)	55782–55812	E	1.4	21	4.9	19
1E 1841−045	11.782898(1)	53824	3.93(1)	53828–53983	E	6.9	0.95	4.7	12
SGR 1900+14	5.19987(7)	53826	9.2(4)	53634–53826	A	7.0	26	0.90	20
1E 2259+586	6.9789484446(4)	51995.6	0.048430(8)	50356–52016	ED	0.59	0.056	230	21
SGR 1801−23	...	...	...	...					
SGR 1808−20	...	...	...	...					
AX J1818.8−1559	...	...	...	...					
AX 1845.0−0258	6.97127(28)	49272	...	...					
SGR 2013+34	...	...	...	...					





Neutron Stars Merger: GW170817



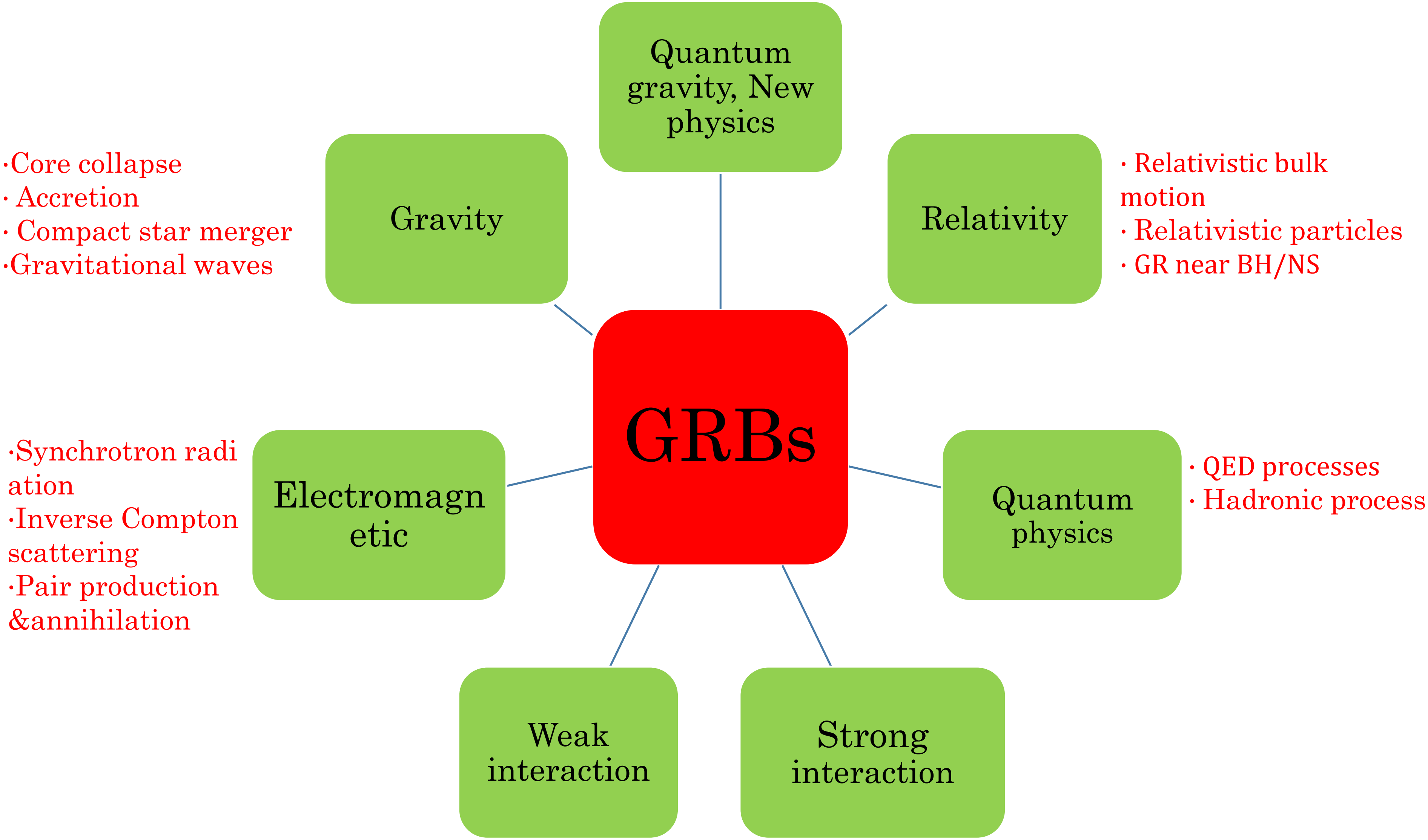


"GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral" [LIGO-Virgo Collaboration, PRL 119 ('17)]; "Multi-messenger Observations of a Binary Neutron Star Merger" [LIGO-Virgo collaboration, Fermi GRB etc, ApJ Lett. 848 ('17)]

Unimaginably strong EM fields are expected: hardly understood!

GRB Astrophysics and Physics

[B. Zhang, *The Physics of GRBs* (CUP, 2019)]



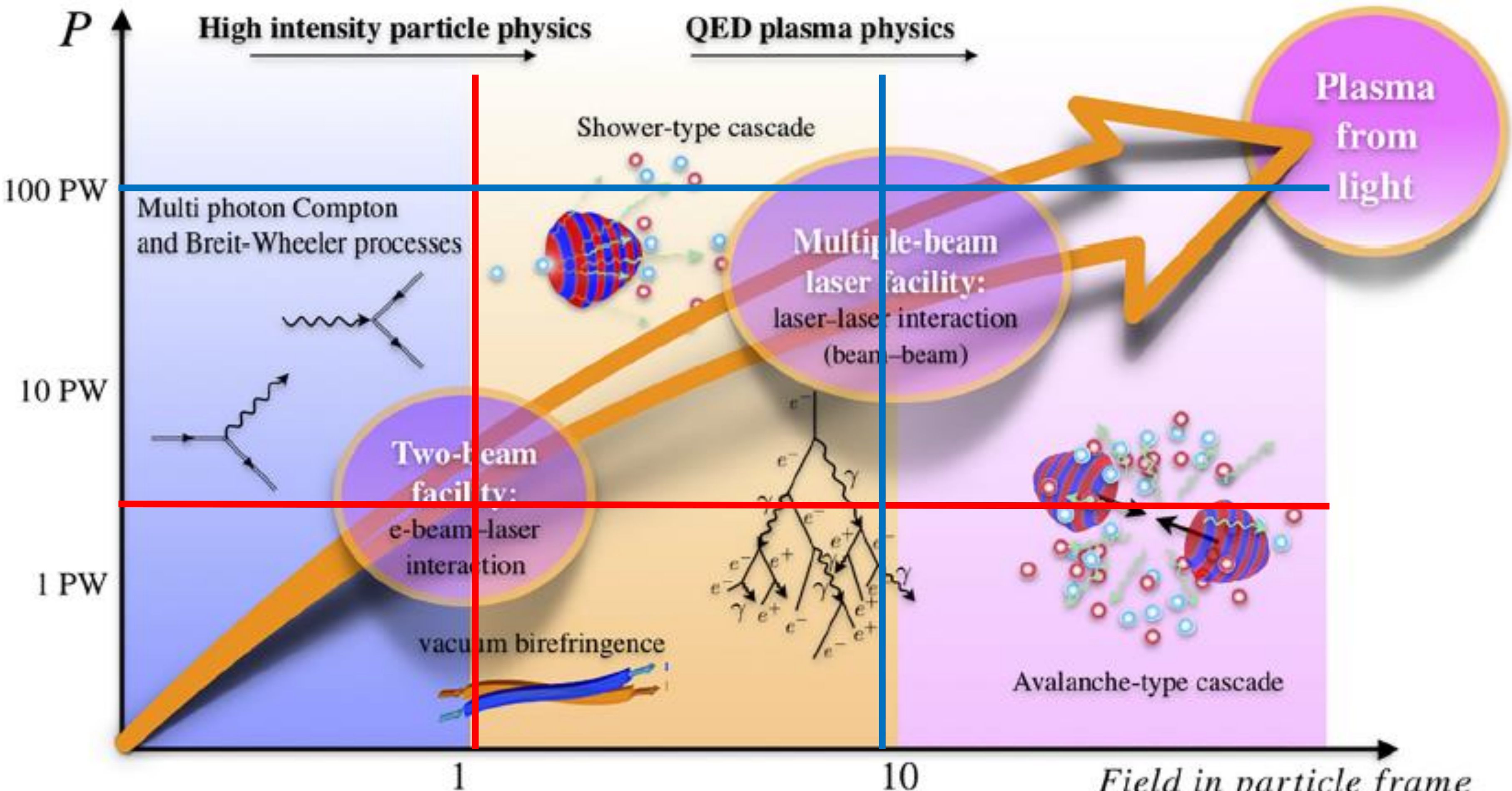
Physics Strong EM Fields

Critical (Schwinger) Field Strength

- Critical electric field E_c and critical magnetic field B_c
  - ✓Energy delivered by field in Compton length ~ electron's rest energy
  - ✓Field energy density ~ electron's rest energy / Compton volume
  - ✓$E_c = \frac{m_e^2 c^3}{\hbar e} = 1.3 \times 10^{16}\ \mathrm{V/cm}$, $B_c = \frac{m_e^2 c^3}{\hbar e} = 4.4 \times 10^{13}\ \mathrm{G}$
  - ✓$I_c = 4.6 \times 10^{29}\ \mathrm{W/cm^2}$
- Vacuum polarization and Schwinger pair production by strong fields
- Physical processes in near-critical or supercritical fields differ from those in weak fields
- Strongest fields on Earth
  - ✓Highest laser intensity: $I = 1.1 \times 10^{23}\ \mathrm{W/cm^2}$ [CoReLS, Optica 8 ('21)] Controlled
  - ✓ATLAS: accelerated Pb ions surrounded by an enormous flux of photons (EM field up to $10^{23}\ \mathrm{V/cm}$) and result in light by light scattering [Nat. Phys. 13 ('17)]: Not controlled

Strong Fields and High Energy Electrons

[Zhang et al, Phys. Plasmas 27 (‘20)]



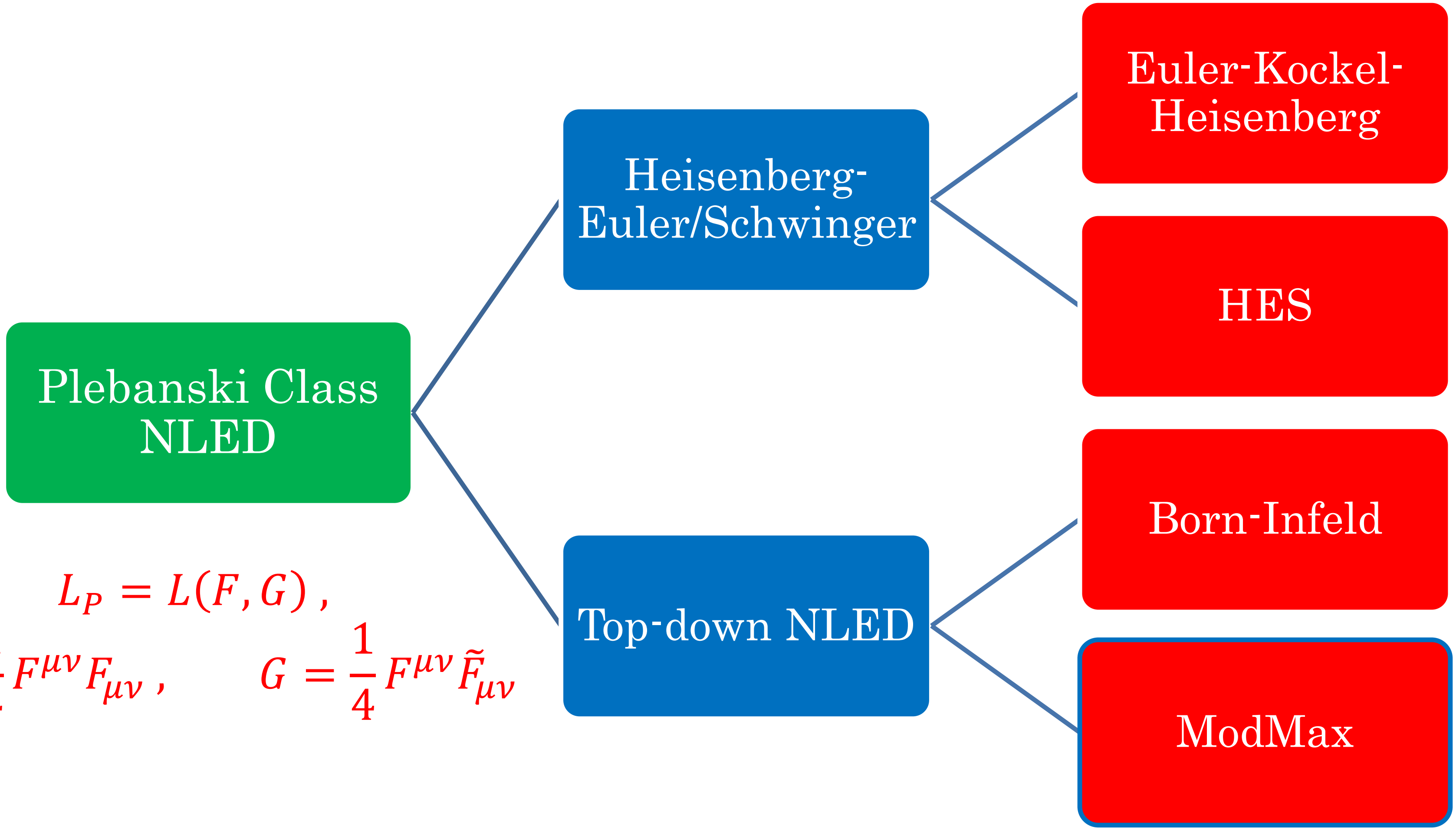
$$\chi = \frac{\textit{Field in particle frame}}{\textit{Critical field}}$$

CoReLS targeting for 2 GeV & 10^{22} $\mathrm{W/cm^2}$

($a_0 = 68,\ \chi = 1.15$)

Nonlinear Electrodynamics

Nonlinear Electrodynamics



$$L_P = L(F, G)\,,$$

$$F = \frac{1}{4} F^{\mu\nu} F_{\mu\nu}\,, \qquad G = \frac{1}{4} F^{\mu\nu} \tilde{F}_{\mu\nu}$$

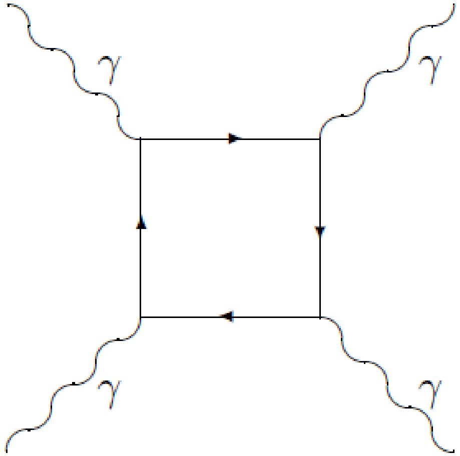
$$L_{MM} = -\cosh\gamma\, F + \sinh\gamma \sqrt{F^2 + G^2}$$

$$L_{BI} = T - \sqrt{T^2 + 2TF - G^2}$$

Vacuum Polarization

Photon virtual pair interaction

Euler-Kockel-Heisenberg

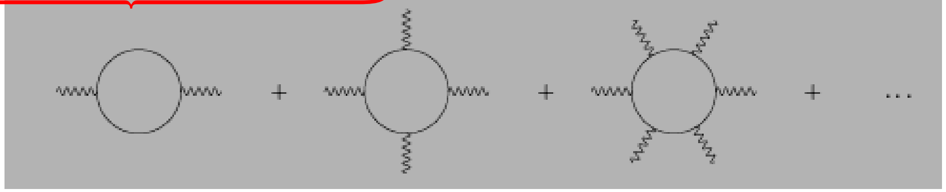


$$\frac{2e^4}{45m^4}[4F^2 + 7G^2]$$

$$F = \frac{1}{2}(\mathrm{B}^2 - \mathrm{E}^2), G = -\mathrm{B} \cdot \mathrm{E}$$

One-Loop Effective Action

charge renormalization



- External legs: EM fields or gravitons
- Internal loops: fermions or bosons

Heisenberg-Euler/Schwinger QED Action

- Maxwell theory and Dirac/Klein-Gordon theory are gauge invariant:

$$F = \frac{1}{4} F^{\mu\nu} F_{\mu\nu} = \frac{1}{2} (\mathrm{B}^2 - \mathrm{E}^2), G = \frac{1}{4} F^{\mu\nu} F^*_{\mu\nu} = -\mathrm{B} \cdot \mathrm{E}$$

$$X = \sqrt{2(F + iG)} = X_r + iX_i$$

- The Heisenberg-Euler/Schwinger effective action per volume and time [Heisenberg-Euler, Z. Phys. 98 ('36); Schwinger, Phys. Rev. 82 ('51)]

$$L^{(1)} = -\frac{1}{8\pi^2} \int_0^\infty ds \frac{e^{-m^2 s}}{s^3} \left[(qs)^2 G \frac{\mathrm{Re} \cosh(qXs)}{\mathrm{Im} \cosh(qXs)} - 1 - \frac{2}{3} (qs)^2 F \right]$$

Worldline formalism for QED

- The propagator of a charge q in an EM field [Feynman, PR 80 ('50); string-inspired formalism, Schubert, PR 355 ('01)]

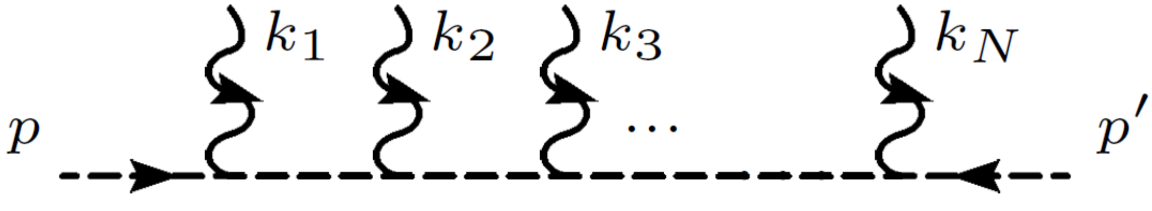
$$D^{xx'}[A] = \left\langle x \right| \int_0^\infty dT \exp[-T(-(\partial - iqA)^2 + m^2)] \left| x' \right\rangle$$

$$= \int_0^\infty dT e^{-m^2 T} \int_{x(0)=x'}^{x(T)=x} \mathcal{D}x(\tau) e^{-\int_0^T d\tau \left(\frac{1}{4}\dot{x}^2 - iq\dot{x}A\left(x(\tau)\right)\right)}$$

- N-plane waves (photons)

$$A_\mu\left(x(\tau)\right) = \sum_{i=1}^{N} \varepsilon_i^\mu \, e^{ik_i\left(x(\tau)\right)}$$

- Dressed propagator after Fourier transforming the end points



Worldline formalism for QED

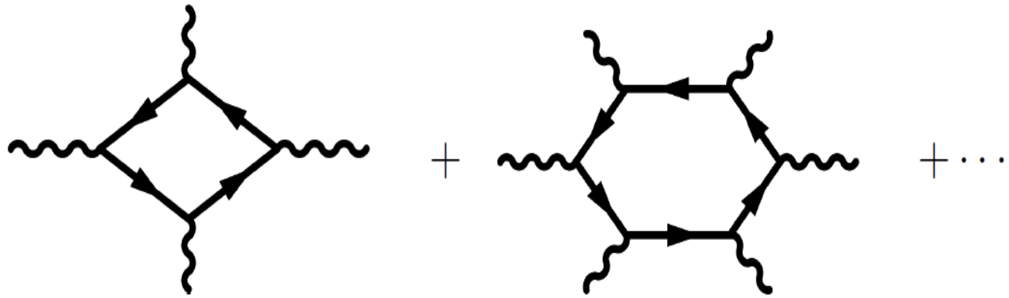
- The one-loop effective action

$$\Gamma[A] = -tr\ln[(-(\partial - iqA)^2) + m^2]$$

$$= \int_0^\infty \frac{dT}{T} tr\exp[-T(-(\partial - iqA)^2 + m^2)]$$

$$= \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int_{x(0)=x(T)} \mathcal{D}x(\tau) e^{-\int_0^T d\tau\left(\frac{1}{4}\dot{x}^2 - iq\dot{x}A\left(x(\tau)\right)\right)}$$

- Equivalent to one-loop N-photon amplitudes



Causality, Unitarity, Regular BHs

- Causality: elementary excitations over the background field should not have a group velocity exceeding the speed of light in vacuum.
- Unitarity: residue of the propagator should not be negative [Shavad, Usov, ('11)]

$$L_P = L(F,G)\,, \qquad F = F^{\mu\nu}F_{\mu\nu}/4\,, \qquad G = F^{\mu\nu}\tilde{F}_{\mu\nu}/4$$

$$L_F > 0\,, \qquad L_{FF} \leq 0\,, \qquad L_F + 2FL_{FF} \geq 0$$

- Einstein-NLED [Bronnikov, 2211.00743]: $\int \sqrt{-g}\big(R - L(F)\big)$
  - $T_{\mu\nu} = -2L_F F_{\mu\alpha}F^{\alpha\nu} + g_{\mu\nu}L/2$
  - Regular BHs: $L \to F$ as $F \to 0$ and $L = finite$ as $F \to \infty$
  - Radial electric field (no duality)
  - Radial (monopole) magnetic fields

Vacuum Polarization, Linear Response

- HES QED action and top-down nonlinear actions belong to the Plebanski class action

$$L_P = L(F, G), \qquad F = F^{\mu\nu}F_{\mu\nu}/4, \qquad G = F^{\mu\nu}\tilde{F}_{\mu\nu}/4$$

- Polarization and magnetization (vacuum polarization)

$$\boldsymbol{D} = \boldsymbol{E} + \boldsymbol{P} = \frac{\delta L(F,G)}{\delta \boldsymbol{E}}, \qquad \boldsymbol{H} = \boldsymbol{B} - \boldsymbol{M} = -\frac{\delta L(F,G)}{\delta \boldsymbol{B}}$$

- Linear response for a probe photon: permittivity, permeability tensors and magneto-electric response

$$\boldsymbol{\delta D} = \epsilon_E \delta \boldsymbol{E} + \epsilon_B \delta \boldsymbol{B}, \qquad \boldsymbol{\delta H} = \overline{\boldsymbol{\mu}}_B \delta \boldsymbol{B} + \overline{\boldsymbol{\mu}}_E \delta \boldsymbol{E}$$

- Magnetoelectric material or multiferroic [Eerenstein et al, Nature 442 ('06), Fiebeig, J Phys. D 38 ('05)]
- Vacuum birefringence (multirefringence) and polarization vectors [Kim, SPK, AIP Conf Proc. ('23), EPJC83,059 ('23), 2210.12890; Sorokin, Fortschr. Phys. 70 ('20)]

Astrophysics in Strong EM Fields

Strong Magnetic Fields in Astrophysics

- Physical upper limit to neutron star magnetic field strength follows from the virial theorem of magnetohydrostatic equilibrium [Chandrasekhar, Fermi, APJ 118 (‘53); Shapiro, Teukosky, BH, WD, NS (‘83)]

$$\frac{4\pi R_{ns}^3}{3}\frac{B^2}{8\pi} \leq G\,\frac{M_{ns}^2}{R_{ns}} \quad \Rightarrow \quad B \leq 10^{18}\left(\frac{M_{ns}}{1.4M_{\odot}}\right)\left(\frac{10\text{ km}}{R_{ns}}\right)^2 G$$

- Dynamo actions [α dynamo: coupling of convective motions and rotation; ω dynamo: differential rotation] in proto-neutron stars can generate magnetic fields of order 10^{15}G or stronger [Thomson, Duncan, APJ 408 (‘93)], and magnetars have been observed [Vasisht, Gotthelf, APJ 486 (‘97)].
- Recent space missions proposed to probe QED regime:
  - ✓Astro2020 Science White Paper, “Magnetars as Astrophysical Laboratories of Extreme QED: The Case for a Compton Telescope,” arXiv:1903.05648
  - ✓“Physics and astrophysics of strong magnetic field systems with eXTP,” [Santangelo et al, Science China, Physics, Mechanics and Astronomy 62 (‘19)]

Laboratory Astrophysics

Vacuum Birefringence on Earth

- The vacuum dressed by a strong field becomes a complicated optical medium.
- Permittivity and permeability tensors in the weak field regime ($^{B}/_{B_c}, ^{E}/_{E_c} \ll 1$) [Klein, Nigam, PR B 135 (‘64)]

$$\varepsilon_{ij} = \delta_{ij}\left[1 + \frac{8\alpha^2}{45m^4}\left(\vec{E}^2 - \vec{B}^2\right)\right] + \frac{28\alpha^2}{45m^4}B_iB_j$$

$$\mu_{ij} = \delta_{ij}\left[1 + \frac{8\alpha^2}{45m^4}\left(\vec{E}^2 - \vec{B}^2\right)\right] + \frac{28\alpha^2}{45m^4}E_iE_j$$

- The vacuum birefringence in a magnetic field

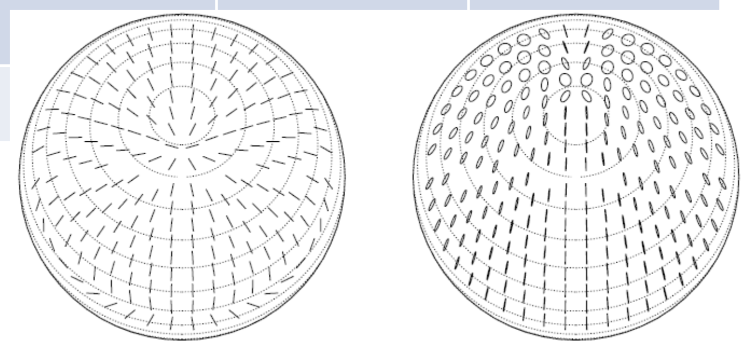
$$n_\perp = 1 + 4\frac{\alpha^2 B^2 \sin\theta^2}{90\pi m^4},\ n_\parallel = 1 + 7\frac{\alpha^2 B^2 \sin\theta^2}{90\pi m^4},\ \ \sin\theta^2 = 1 - \left(\vec{k}\cdot\frac{\vec{B}}{B}\right)^2$$

- Project for the experiments with ultra-intense laser (driver) and x-ray free electron laser (probe) [SEL 100-PW laser project, Shen et al, PPCF 60, 2018]

Vacuum Birefringence in Highly Magnetized Neutron Stars/Magnetars

X-Ray Polarimetry (relative phase shift $\Delta\phi \sim \frac{L}{\lambda}\left(\frac{B}{B_c}\right)^2$ & ellipticity)

	Radius [cm]	Magnetic Field [G]	μ [G/cm^3]	r_{pl}	r_{pl}/R	t_{obs}
Magnetar	10^6	10^{15}	10^{33}	$3.0\text{x}10^8$	300	10ks
XDINS X-ray dim isolated NS	10^6	10^{13}	10^{31}	$4.7\text{x}10^7$	50*	1ks
XRP X-ray pulsar	10^6	10^{12}	10^{30}	$1.9\text{x}10^7$	20	100ks
msXRP	10^6	10^9	10^{27}	$1.2\text{x}10^6$	1.2	
AM Her	10^6	10^8	10^{35}	$1.9\text{x}10^9$	1.9	
Black Hole	10^{6+}	?	N/A			

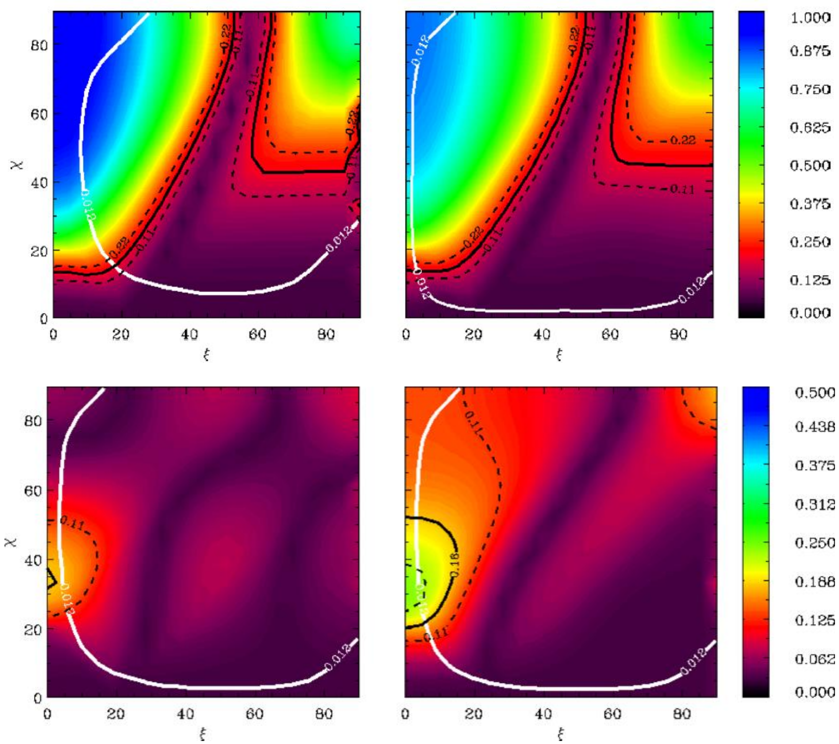


[Heyl, Caiazzo, Galaxies 6 ('18)] "Strongly Magnetized Sources: QED and X-ray Polarization"

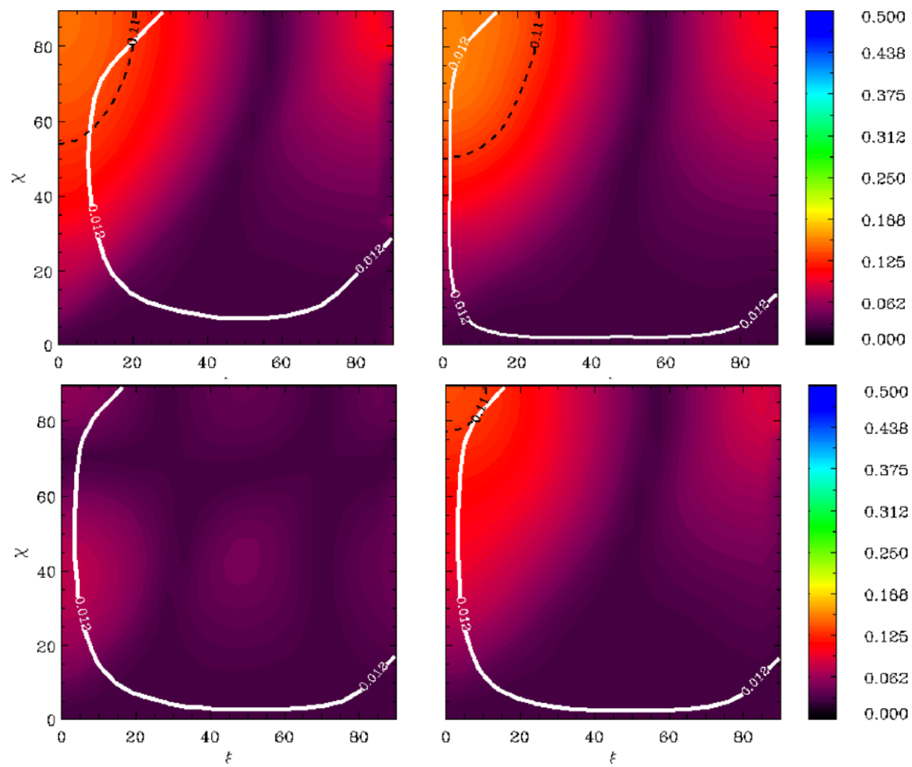
Evidence of Vacuum Birefringence from NS RX J1856.5-3754

[Mignani et al, MNRAS 465 ('17)]

With vacuum polarization effects



Without vacuum polarization effects

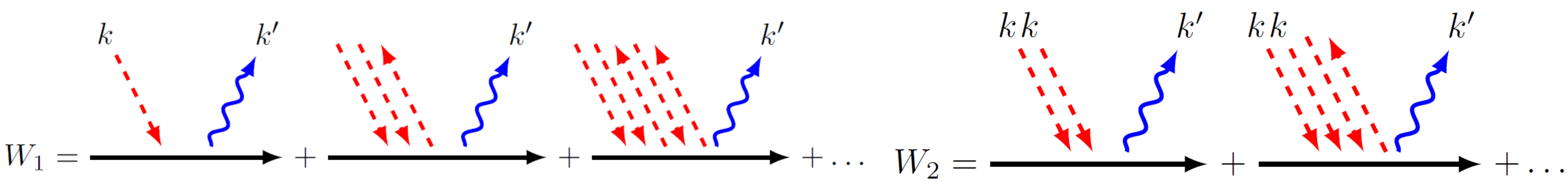


Strong Field QED Processes

Nonlinear Compton Scattering (I)

- Dressed propagator for electron:

$$S_{i\to f} = -ie \int d^4x \overline{\Psi}_{p'} \gamma^\mu \varepsilon^*_\mu(l) e^{ilx} \Psi_p = (2\pi)^4 \sum_{s\geq 1} M^{(s)}\, \delta^4(p' + k' - p - sk)$$



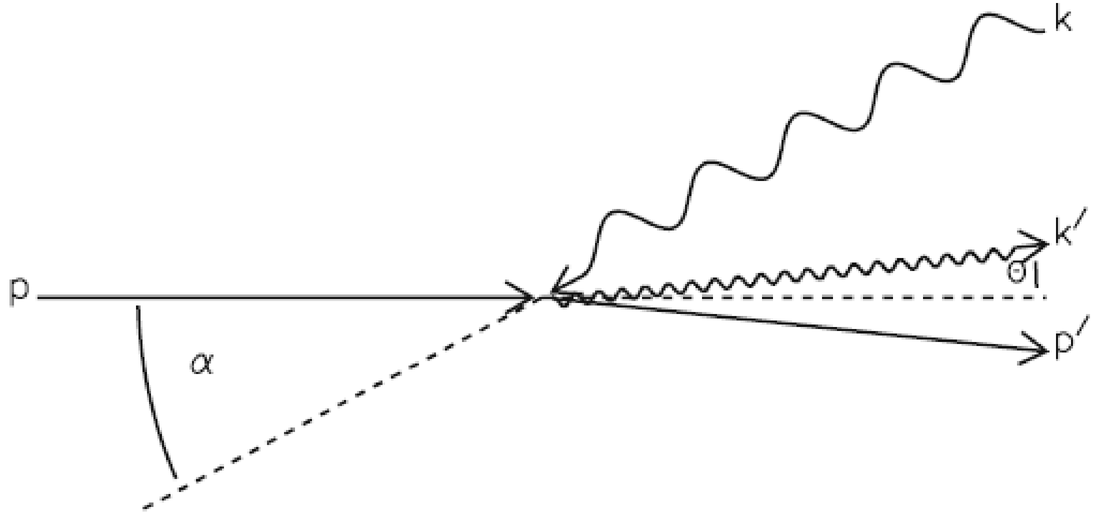
- Narozhny-Nikishov-Ritus formula for photon number n [(‘64)]

$$W^{(n)}_{i\to f}(a_0, \chi) = \frac{\alpha m^2}{4q_0} \int_0^{u_s} \frac{du}{(1+u^2)^2} \left[-4J_n^2(z) + a_0^2 \left(2 + \frac{u^2}{1+u} \right) \left(J_{n+1}^2(z) + J_{n-1}^2(z) - 2J_n^2(z) \right) \right]$$

$$z = \frac{a_0^2 \sqrt{1+a_0^2}}{\chi} \sqrt{u(u_n - u)}\,,\ u_n = \frac{2n\chi}{a_0(1+a_0^2)},\ \chi = \frac{e[\hbar]}{m^3[c^4]} \sqrt{-\left(F_{\mu\nu} p^\nu\right)^2}$$

Nonlinear Compton Scattering (II)

- SLAC- E144 scattering schematics [PRD60 ('99)]:

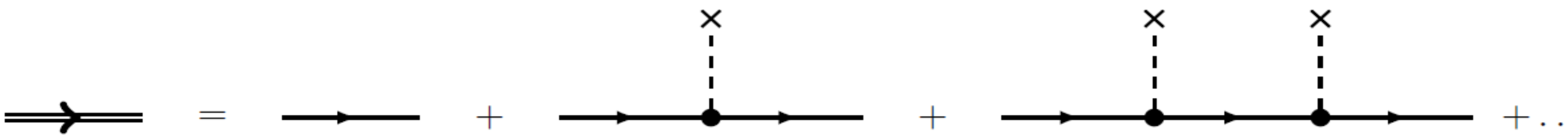


- Kinematic relation for $p + nk \rightarrow p' + k'$ [Melissinos, in *Strong Field Laser Physics* ('08)]:

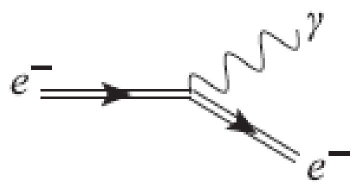
$$\omega' = \frac{2n\omega\gamma^2(1+\beta\cos\alpha)}{2\gamma^2(1-\beta\cos\theta) + \left(\frac{2n\omega\gamma^2}{\gamma mc^2} + \frac{\eta^2}{1+\beta\cos\alpha}\right)(1+\cos(\theta-\alpha))}$$

$$\eta = \frac{e}{m}\sqrt{\left|\langle A^\mu A_\mu \rangle\right|} = \frac{eE_{rms}}{m\omega c} = \frac{a_0}{\sqrt{2}}$$

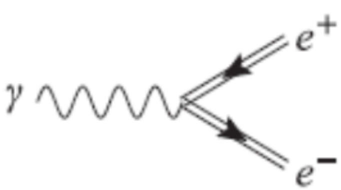
Nonlinear Compton Scattering

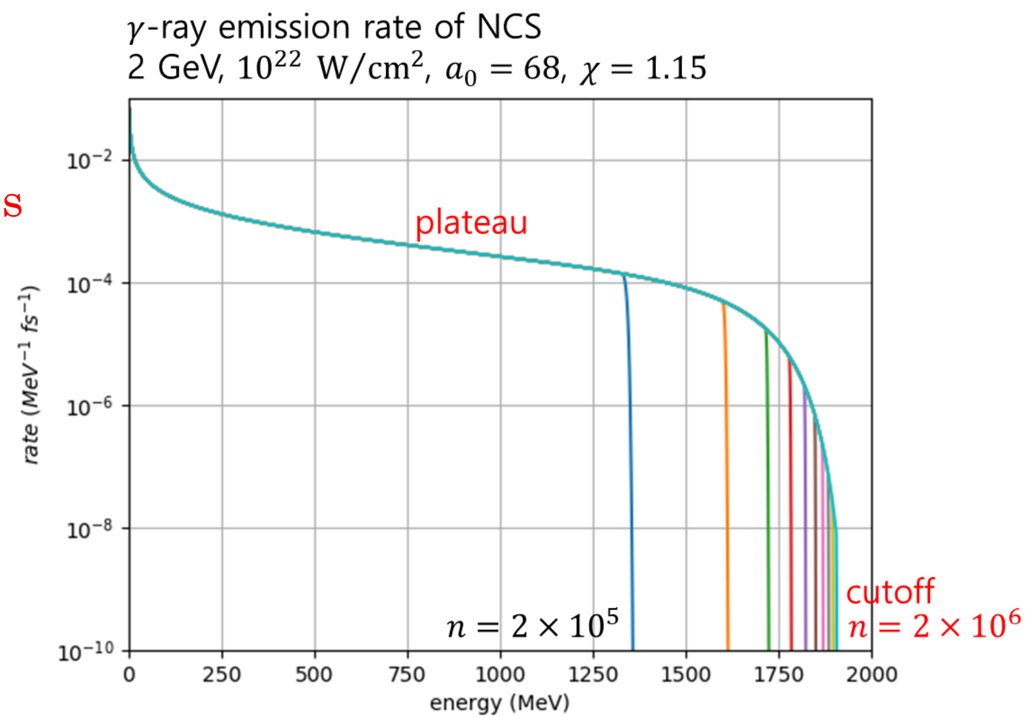


Unlike a free electron, an electron dressed by a strong field can spontaneously emits an energetic photon.
Non-perturbative features: high rate, long plateau, extreme order



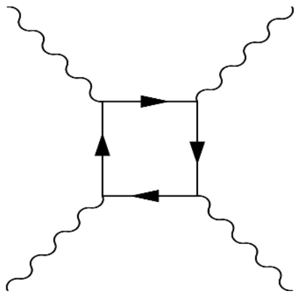
The lowest-order SFQED process
The seed for other SFQED processes, e.g.,
nonlinear Breit-Wheeler pair production and QED cascades





Photon-Photon Scattering

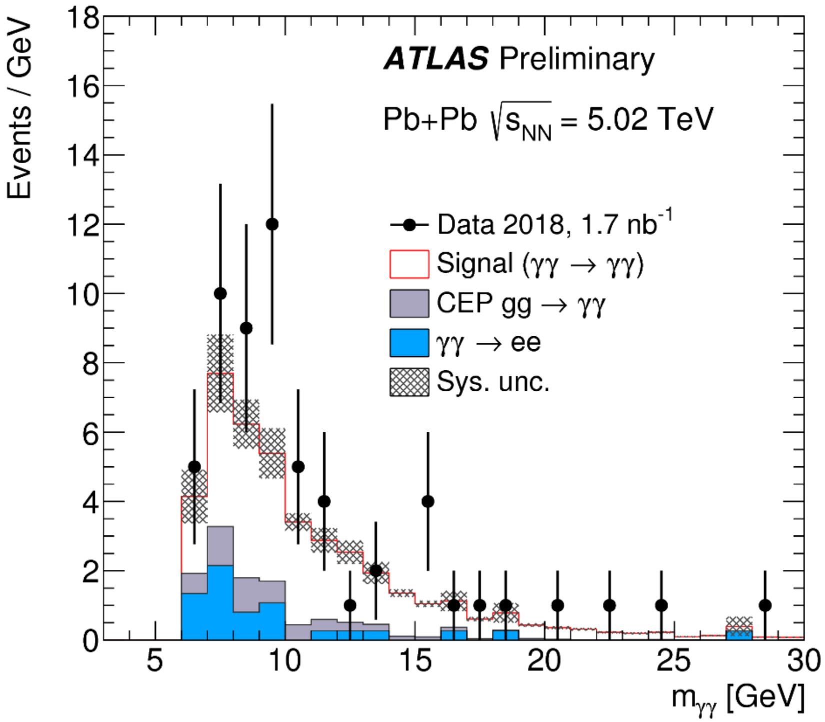
- Classical Maxwell theory is linear and thus prohibits a self-interaction (direct γ-γ scattering).
- QED permits γ and γ to interact with virtual e-e+ pair from the Dirac sea: the cross section in the low energy limit of the two colliding γs in the center of momentum frame) [Euler ('36); Akhiezer ('37); Karplus, Neuman ('50)]: a vacuum polarization effect



$$\sigma_{\gamma\gamma\to\gamma\gamma} = \frac{973}{81000\pi}(\alpha^2\lambda_C)^2\eta^3, (\eta = \frac{2\omega^{*2}}{m^2})$$
$$= 7.4 \times 10^{-66}(\omega^*[eV])^6$$

- ELI is highly likely to detect it. ATLAS collaboration of LHC has observed the photon-photon scattering ('15) and confirmed by CMS.

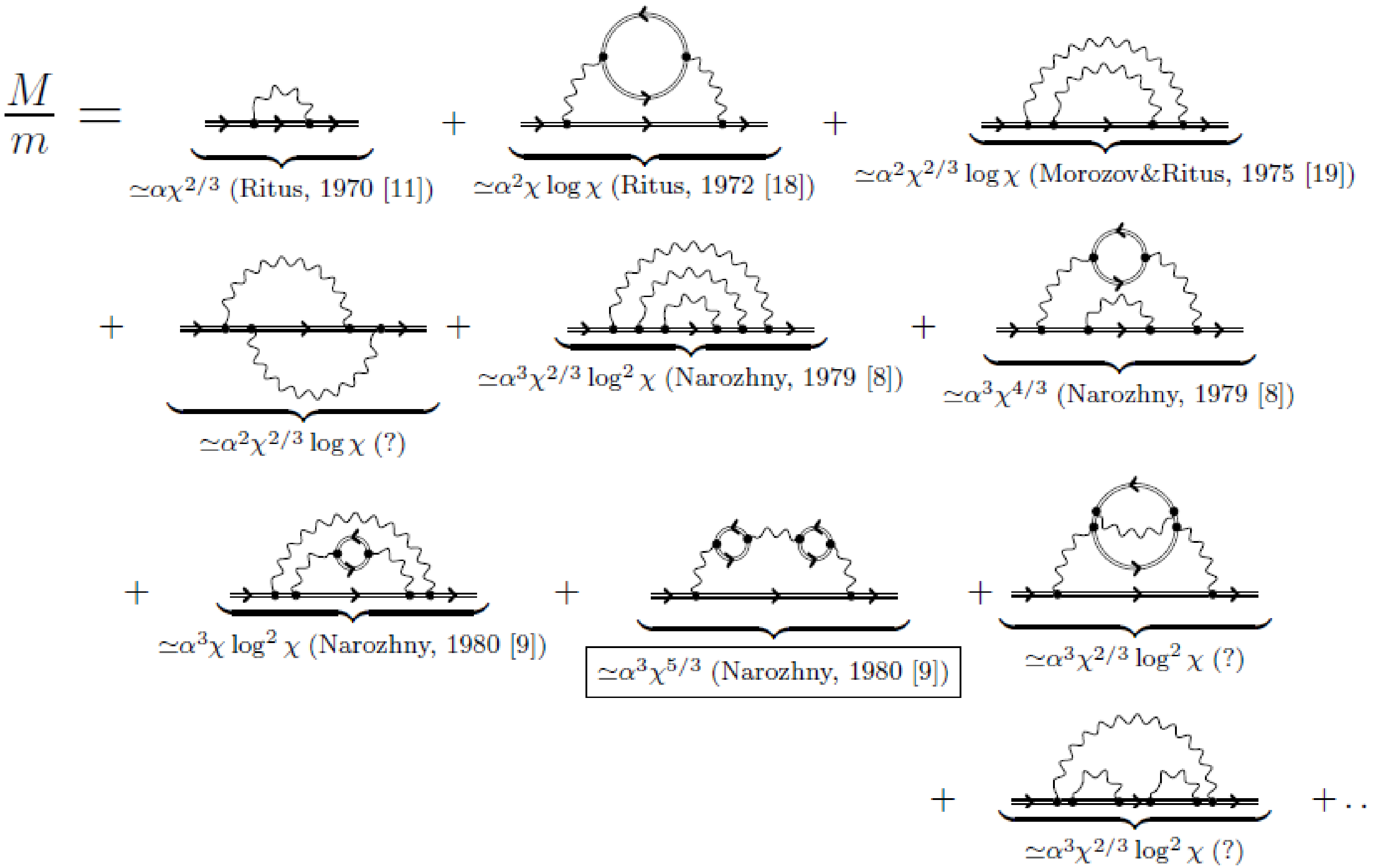
Light by Light Scattering at ATLAS



- Pb ions accelerated to very hig h energy are surrounded by an enormous flux of photons.
- EM field up to 10^{23} V/cm.
- Two lead ions pass close by ea ch other at the center ATLAS and result in light by light sca ttering [Nat. Phys. 13 ('17)].

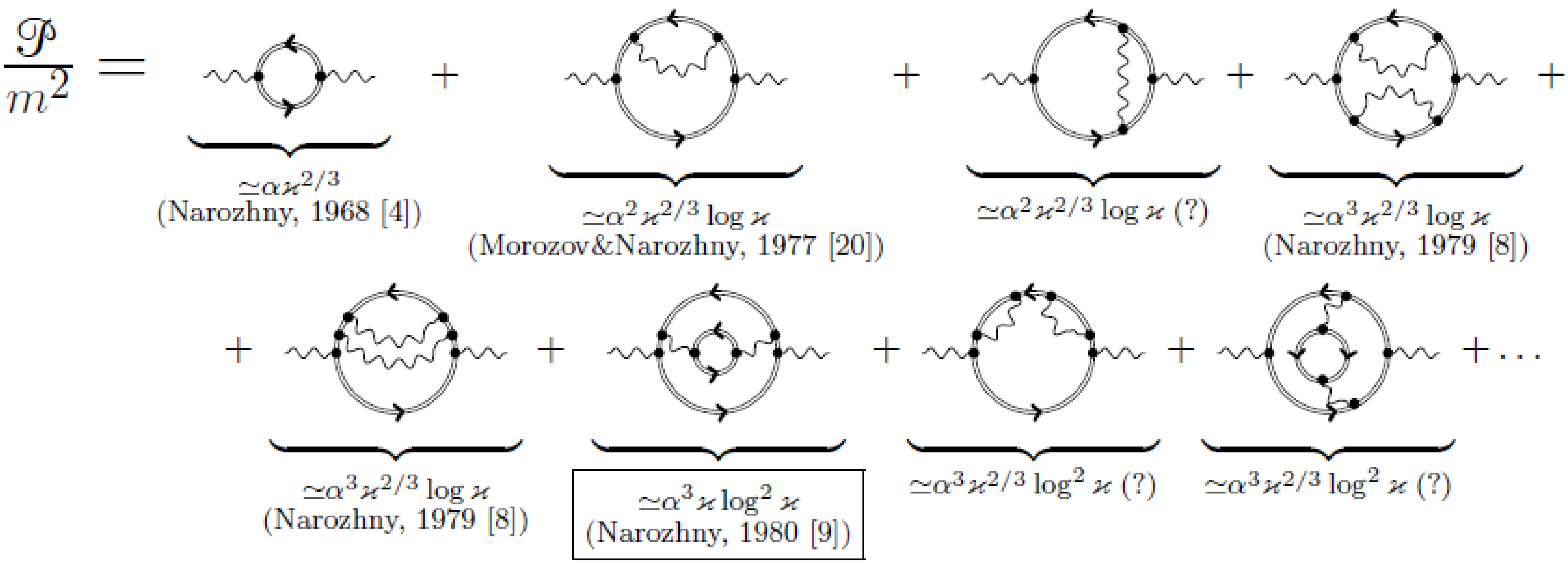
Ritus-Narozhny Conjecture (I)

- Perturbative QED breaks down at $\alpha\chi^{2/3}$ due to radiative corrections:



Ritus-Narozhny Conjecture (II)

- Radiative corrections to polarization:



Ritus-Narozhny Conjecture (III)

- Two different parameters for high intensity and energy [Ilderton, PRD 99 ('19)]:

$$a_0\,, \qquad b = \frac{kp}{m^2}$$

- Radiative corrections in high field

$$\underbrace{P(e,\gamma|e)}_{\chi\sim a_0\to\infty} \sim \frac{\alpha\chi^{2/3}}{b}$$

- Radiative corrections in high energy

$$\underbrace{P(e,\gamma|e)}_{\chi\sim b\to\infty} \sim \frac{\alpha a_0^3}{\chi}\ln(\chi)$$

Magnetic Monopoles

Dirac Monopoles in Maxwell Theory

- Dirac introduced a magnetic monopole (& Dirac string) that quantizes charges ($eg = n/2$) and symmetrizes the Maxwell theory under $\vec{E} \leftrightarrow \vec{B}$:

$$\nabla \cdot \vec{E} = 4\pi\rho_e \,, \qquad \nabla \cdot \vec{B} = 4\pi\rho_m$$

$$\nabla \times \vec{E} = -4\pi\vec{j}_m - \partial\vec{B}/\partial t, \qquad \nabla \times \vec{B} = 4\pi\vec{j}_e + \partial\vec{E}/\partial t$$

- Dirac monopole is Wu-Yang monopole in fiber bundle topology.
- Lorentz force on dyons (electric charge q and magnetic charge g):

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) + g\left(\vec{B} - \vec{v} \times \vec{E}\right)$$

- EM field generated by a point dyon

$$\vec{E} = q\frac{\vec{r}}{r^3} \,, \qquad \vec{B} = g\frac{\vec{r}}{r^3}$$

‘t Hooft-Polyakov Monopole

- Nonabelian Wu-Yang monopole in $SU(2)$: $A_\mu = A_\mu^{Dirac}\sigma_3/2$
- ‘t Hooft-Polyakov monopole in $SU(3)$ [‘t Hooft, NPB 79 (‘74); Polyakov, JETP 20 (‘74)]

$$L = -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} + \frac{1}{2}(D^\mu \phi^a)(D_\mu \phi^a) - V(\phi)$$

$$\phi^a = \frac{r^a}{er^2}H(evr), \qquad A_n^a = \epsilon_{amn}\frac{r^m}{er^2}[1 - K(evr)], \qquad A_0^a = 0$$

- Magnetically charged black hole in $SU(2)$ and the end state of an unstable RN black hole when the horizon radius larger than that of RN black hole with the same mass [Lee, Nair, Weinberg, PRL 68 (‘92)].

Conclusion

- Schwinger Effect in Curved spacetimes
  - Symmetric spacetimes T^2, S^2, H^2. BUT general spacetimes?
- BHs in dS space
  - Magnetogenesis (QED in dS): cosmological magnetic fields (amplification of seed B)
  - Breitenlohner-Freedman bound for stability (near-) extremal RN or KN BHs
  - Primordial BHs with magnetic monopoles (dark matter?)
  - Primordial BHs with (dark) electric and/or magnetic charges (dark matter?)
  - EM and GW radiations from PBH binaries with (dark) charges (observations?) [Lang Liu et al, ('20), ('22), ('23)]
- X-ray polarimetry for highly magnetized sources: eXTP, Compton Telescope,
- Laboratory Astrophysics: CoReLS, SEL, ELI